13. Decoherence and Consistent Histories

I. Quantum Probabilities and Interference

Basic Idea: Classical probability theory is based on classical (Boolean) logic. The probabilities defined by the Born Rule in quantum mechanics are based on quantum (non-Boolean) logic. In particular, they do not satisfy the classical or-addition axiom of classical probability theory.

Classical Probability Theory and the Classical Or-Addition Rule

Basic Idea: Begin with a set $\Omega$ of simple events (the sample space). Then form a collection $\mathcal{F}$ of compound events by taking all possible (classical) logical combinations of the simple events. Then define a probability function $Pr_c$ that maps elements $\mathcal{F}$ to the real interval $[0, 1]$. Finally slap on a bunch of axioms $Pr_c$ must satisfy.

So: A classical probability theory is given by a triple: $(\Omega, \mathcal{F}, Pr_c)$.

The probability function $p$ is required to satisfy the following axioms:

\begin{align*}
(C1) & \quad Pr_c(\emptyset) = 0 \\
(C2) & \quad Pr_c(\neg A) = 1 - Pr_c(A) \\
(C3) & \quad Pr_c(A \cup A') = Pr_c(A) + Pr_c(A') - Pr_c(A \cap A')
\end{align*}

(C3) is the Classical Or-Addition Rule. It says, "The probability of either $A$ or $A'$ occurring is equal to the sum of the probability of $A$ occurring and the probability of $A'$ occurring, minus the probability of both $A$ and $A'$ occurring." (If $A$ and $A'$ are mutually exclusive, then this latter probability is zero.)

Example: Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ represent the simple events corresponding to all possible results of a single roll of a die. Now form $\mathcal{F}$ by taking all possible classical logical combinations of these events:

"1", "2", ..., etc., "1 or 2", "1 or 3", ...., etc., "not 1", "not 2", etc...

These correspond to the sets:

\begin{align*}
\{1\}, \{2\}, \ldots, \text{etc.}, \{1\} \cup \{2\}, \{1\} \cup \{3\}, \ldots, \text{etc.}, \neg\{1\}, \neg\{2\}, \text{etc}.
\end{align*}

\begin{align*}
\{1, 2\} & \quad \{2, 3, 4, 5, 6\}
\end{align*}
Quantum Probability Theory and "Interference"

Basic Idea: Replace the classical sample space set $\Omega$ with a Hilbert space $H$. Now form a collection $\mathcal{L}$ of subspaces of $H$ (the "compound events") by taking all possible quantum logical combinations of the rays in $H$ (the simple events). Define the probability function $Pr_q$ on them by the Born Rule.

So: A quantum probability theory is given by a triple: $(H, \mathcal{L}, Pr_q)$, where $Pr_q$ is defined by

$$Pr_q(|a_i\rangle, |\psi\rangle) = |\langle a_i | \psi \rangle|^2,$$

where $|a_i\rangle, |\psi\rangle$ are elements of $H$.

Recall: This is the probability that the system possesses the value $a_i$ of a property represented by an operator with eigenvector $|a_i\rangle$, when the system is in the state represented by $|\psi\rangle$.

Main Result: Quantum probabilities, so-defined, do not in general satisfy C3! They do satisfy the following (where $V$, $W$ are subspaces of $\mathcal{H}$ and $0$ is the "zero" subspace):

\begin{align*}
(Q1) \quad & Pr_q(0) = 0 \\
(Q2) \quad & Pr_q(V^\perp) = 1 - Pr_q(V) \\
(Q3) \quad & Pr_q(V \oplus W) = Pr_q(V) + Pr_q(W), \quad \text{when } V \perp W
\end{align*}

Recall: Linear span $\oplus$ does not correspond to classical "or".

Finally, define the probability function to be $Pr_r(\{i\}) = 1/6$, for $i = 1...6$.

Then, for instance, the probability of getting either 1 or 3 on a single roll is given by:

$$Pr_r(\{1\} \cup \{3\}) = Pr_r(\{1\}) + Pr_r(\{3\}) - Pr_r(\{1\} \cap \{3\}) = 1/6 + 1/6 - 0 = 1/3$$

The probability of getting either a value in the range $\{1, 2, 3\}$ or a value in the range $\{3, 4, 5\}$ on a single roll is:

$$Pr_r(\{1, 2, 3\} \cup \{3, 4, 5\}) = Pr_r(\{1, 2, 3\}) + Pr_r(\{3, 4, 5\}) - Pr_r(\{1, 2, 3\} \cap \{3, 4, 5\}) = [Pr_r(\{1\}) + Pr_r(\{2\}) + Pr_r(\{3\})] + [Pr_r(\{3\}) + Pr_r(\{4\}) + Pr_r(\{5\})] - Pr_r(\{3\}) = [1/6 + 1/6 + 1/6] + [1/6 + 1/6 + 1/6] - 1/6 = 5/6$$

ASIDE: Recall from the lecture on quantum logic that the collection of subspaces $\mathcal{L}$ is not a Boolean algebra. Technically, it forms what’s called a lattice, and is obtained by closing $H$ (thought of as a collection of rays) under the operations of orthocomplement and linear span.
**Example:** 2-slit probabilities and interference

With both slits open, experiments indicate that the probability that \( e \) is located at \( x \) is given by

\[
|\psi_A(x)|^2 + |\psi_B(x)|^2.
\]

This is **not** equal to the sum \(|\psi_A(x)|^2 + |\psi_B(x)|^2\), which, according to (C3), represents the probability that the electron *either* went through slit \( A \) or slit \( B \) (assuming these are mutually exclusive events).

**ASIDE:** The Born Rule tells us that the state corresponding to the probability distribution \(|\psi_A(x) + \psi_B(x)|^2\) is the superposition \(\psi_A(x) + \psi_B(x)\). This is in the subspace \(V \oplus W\) which is the linear span of the subspace \(V\) containing the state \(\psi_A(x)\) and the subspace \(W\) containing the state \(\psi_B(x)\).
Let's see how this works in the more general setting of projection operators.

**First Recall:**

1. The projection operator $P_{|a_i\rangle} = |a_i\rangle\langle a_i|$ corresponds to the 1-dim subspace defined by $|a_i\rangle$ (i.e., the ray in which $|a_i\rangle$ is pointing).
2. $\sum_i P_{|a_i\rangle} = 1.$

**Now Consider:** The "trace" of an operator.

**Def.** Suppose $Q$ is a linear operator on an $N$-dim vector space $\mathcal{H}$ with orthonormal basis $|b_1\rangle, |b_2\rangle, \ldots |b_N\rangle$. Then the **trace** $\text{Tr}(Q)$ of $Q$ is given by:

$$\text{Tr}(Q) \equiv \sum_{i=1}^{N} \langle b_i | Q | b_i \rangle$$

**Note:** The trace is just the sum of the diagonal elements of any matrix representation of $Q$. It turns out that all such representations have this sum in common! So the trace is independent of the basis it’s calculated in.

The trace has the following properties:

- $\text{Tr}(\lambda A) = \lambda \text{Tr}(A)$ where $\lambda$ is any number
- $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$
- $\text{Tr}(AB) = \text{Tr}(BA)$

**The Born Rule can now be rewritten in terms of projection operators:**

$$\text{Pr}_q(\text{value of A is } a_i \text{ in state } |\psi\rangle) = |\langle a_i | \psi \rangle|^2$$

$$= \langle \psi | a_i \rangle \langle a_i | \psi \rangle$$

$$= \sum_j \langle \psi | P_{|a_j\rangle} | a_i \rangle \langle a_i | \psi \rangle$$

where $\sum_j P_{|a_j\rangle} = 1$

$$= \sum_j \langle \psi | a_j \rangle \langle a_j | a_i \rangle \langle a_i | \psi \rangle$$

$$= \sum_j \langle a_j | a_i \rangle \langle a_i | \psi \rangle \langle \psi | a_j \rangle$$

$$= \text{Tr}(|a_i\rangle \langle a_i | \psi \rangle \langle \psi | a_i\rangle)$$

$$= \text{Tr}(P_{|a_i\rangle} P_{|\psi\rangle})$$

**SO:**

$$\text{Pr}_q(\text{value of A is } a_i \text{ in state } |\psi\rangle) = \text{Tr}(P_{|a_i\rangle} P_{|\psi\rangle})$$
**Again:** $P_{|a_i\rangle}$ is the projection operator corresponding to the state $|a_i\rangle$ (more precisely, the 1-dim subspace defined by $|a_i\rangle$). $P_{|ψ\rangle}$ is the projection operator corresponding to the state $|ψ\rangle$ (more precisely, the 1-dim subspace defined by $|ψ\rangle$).

**Terminology:** The projection operator corresponding to a state is called the **statistical operator** (alternatively, **density matrix**) for the state.

**Aside on terminology:** Suppose the exact state of a system is known only to fall within some set $\{ |ψ_k\rangle \}$ to which weights $w_k$ can be assigned such that $\sum w_k = 1$. Then the statistical operator $W$ for the system is defined by $W = \sum w_k P_{|ψ_k\rangle}$ (this need not be a projection operator). A pure state is one for which $W$ reduces to a single term. This is the case in which the exact state of the system is known, say $|ψ\rangle$, so the statistical operator is just the projection operator $P_{|ψ\rangle}$. A mixed state is one for which $W$ has more than one term.

**Now:** Consider the composite state $m$-e of a Hardness measuring device and a black electron:

$$|ψ\rangle = \sqrt{\frac{1}{2}} \{ |\text{"hard"}_m|\text{hard}_e\rangle + |\text{"soft"}_m|\text{soft}_e\rangle \}$$

call it simply $\sqrt{\frac{1}{2}} \{ |\text{"hard"}|h\rangle + |\text{"soft"}|s\rangle \}$

It's statistical operator $P_{|ψ\rangle} = |ψ\rangle\langleψ|$ is given by:

$$P_{|ψ\rangle} = \frac{1}{2} \{ |\text{"hard"}\rangle\langle\text{hard}| + |\text{"soft"}\rangle\langle\text{soft}| \} \{ (|\text{"hard"}\rangle\langle\text{hard}| + |\text{"soft"}\rangle\langle\text{soft}| \}$$

$$= \frac{1}{2} \{ |\text{"hard"}\rangle\langle\text{hard}| + |\text{"soft"}\rangle\langle\text{soft}| \} \{ (|\text{"hard"}\rangle\langle\text{hard}| + |\text{"soft"}\rangle\langle\text{soft}| \}$$

$$= \frac{1}{2} \{ P_{|\text{"hard"}_m\rangle\langle\text{hard}_e|} + P_{|\text{"soft"}_m\rangle\langle\text{soft}_e|} \} + |\text{"hard"}\rangle\langle\text{hard}| |\text{"soft"}\rangle\langle\text{soft}|$$

**So:**

$$\text{Pr}_q(\text{value of } A \text{ is } a_i \text{ in state } |ψ\rangle) = \text{Tr}(P_{|a_i\rangle} P_{|ψ\rangle})$$

$$= \text{Tr}(\frac{1}{2} P_{|a_i\rangle} P_{|\text{"hard"}_m\rangle\langle\text{hard}_e|} + \frac{1}{2} P_{|a_i\rangle} P_{|\text{"soft"}_m\rangle\langle\text{soft}_e|}) + \text{Tr}(\frac{1}{2} P_{|a_i\rangle} |\text{"hard"}\rangle\langle\text{hard}| |\text{"soft"}\rangle\langle\text{soft}|)$$

$$+ \text{interference terms}$$
**What this means:**
The probability associated with the superposed state is not in general the sum of the probabilities associated with its terms. There are "interference terms" that are not in general zero. Quantum probabilities do not obey the classical or/addition rule (C3).

**II. Decoherence and Interference**

**Claim:** Interference effects are destroyed when a system interacts with its environment. In particular, when an observer ends up in an entangled state with a measuring device, environmental interactions effectively "decohere" the state into one in which the observer effectively records a definite measurement outcome.

**Basic Argument:**

It is experimentally impossible to distinguish between:

1. The state:
   \[ \sqrt{\frac{1}{2}}|\text{"hard"}_m\text{hard}_e\rangle_E + \sqrt{\frac{1}{2}}|\text{"soft"}_m\text{soft}_e\rangle_E \]
   No collapse: no definite outcome.

2. Either of the states:
   \[ |\text{"hard"}_m\text{hard}_e\rangle_E \text{ or } |\text{"soft"}_m\text{soft}_e\rangle_E \]
   The result of collapse: a definite outcome.

where \( |\text{hard}_E\rangle, |\text{soft}_E\rangle \) are states of the environment \( E \) in which it is correlated with a hard electron and a soft electron, respectively.

**Recall:** To distinguish between (1) and (2), we would need a very complex multi-particle property that (1) possesses and that neither state in (2) possesses. Given that \( E \) realistically has a huge number of degrees of freedom, it is experimentally impossible to measure such a property. So (1) and (2) are indistinguishable for all practical purposes!

**What this is supposed to mean:**

Whenever the post-measurement state of a composite system is of the form of (1), it does, for all practical purposes, describe a situation in which a definite measurement outcome occurred. The environment, for all practical purposes, collapses the entangled superposition.
Let's see how this works using statistical operators.

The statistical operator \( P_{\psi} = |\psi\rangle\langle\psi| \) for the state in (1) is:

\[
P_{\psi} = \frac{1}{2}\{ |h^n\rangle h| E_h \rangle + |s^n\rangle s| E_s \rangle \} \{ |h^n\rangle h\langle E_h | + |s^n\rangle s\langle E_s | \}
\]

\[
= \frac{1}{2}\{ |h^n\rangle h| E_h \rangle \langle h^n| E_h | + |s^n\rangle s| E_s \rangle \langle s^n| E_s | \} \\
+ |h^n\rangle h| E_h \rangle \langle s^n| E_s | + |s^n\rangle s| E_s \rangle \langle h^n| E_h | \}
\]

\[
= \frac{1}{2}\{ |h^n\rangle \langle h^n| E_h \rangle + |s^n\rangle \langle s^n| E_s \rangle \} \\
+ \frac{1}{2}|s^n\rangle \langle s^n| E_s \rangle \langle h^n| E_h | \}
\]

\[
= \frac{1}{2} P_{\text{hard}^n} \otimes P_{|E_h \rangle} + \frac{1}{2} P_{\text{soft}^n} \otimes P_{|E_s \rangle} \\
+ \frac{1}{2} |s^n\rangle \langle s^n| E_s \rangle \langle h^n| E_h | \}
\]

\[
\text{statistical operator for } \langle \text{"hard"} \rangle \text{\_statistical} \] \\
\text{statistical operator for } \langle \text{"soft"} \rangle \text{\_statistical}
\]

\[
\text{interference terms!}
\]

Now: Take the "partial trace" of \( P_{\psi} \) with respect to the Environment basis \( |E_h \rangle, |E_s \rangle \):

\[
\text{Tr}_E(P_{\psi}) = \langle E_h | P_{\psi} | E_h \rangle + \langle E_s | P_{\psi} | E_s \rangle \\
= \frac{1}{2} P_{\text{hard}^n} \otimes P_{|E_h \rangle} + \frac{1}{2} P_{\text{soft}^n} \otimes P_{|E_s \rangle}
\]

Aside: This is called a "partial trace" because we are not using a "full" basis to calculate the diagonal elements of \( P_{\psi} \). The "full" basis should be a "3-particle" basis like \( |h^n\rangle h| E_h \rangle, |s^n\rangle s| E_s \rangle, |h^n\rangle h| E_s \rangle, |h^n\rangle s| E_h \rangle \), etc (eight such basis vectors). In this case, we're only using a "1-particle" basis \( |E_h \rangle, |E_s \rangle \). This particular "partial trace" results, not in a number, but a 2-particle operator!

Result: "Tracing over the environment" kills the interference terms: They include the operators \( |E_h \rangle \langle E_s | \) and \( |E_s \rangle \langle E_h | \). When you slap \( |E_h \rangle \) or \( |E_s \rangle \) to both sides of these operators, you get zero! When you do the same to the operators \( P_{\text{hard}^n} \otimes P_{|E_h \rangle} \) and \( P_{\text{soft}^n} \otimes P_{|E_s \rangle} \), you just kill the Environment operators at the end. What we're left with is just the statistical operator for the states \( |\text{"hard"} \rangle \text{\_statistical} \) or \( |\text{"soft"} \rangle \text{\_statistical} \).

Basic Claim: "Tracing over the environment" is the mathematical expression of having the environment decohere the entangled superposed state in (1).
**Does decoherence solve the measurement problem? NO!**

When we "trace out the environment", we're left with the statistical operator

\[ \frac{1}{2} P_{|\text{"hard"}\rangle_m |\text{hard}\rangle_e} + \frac{1}{2} P_{|\text{"soft"}\rangle_m |\text{soft}\rangle_e} \]

This is the statistical operator for a "mixed state". This is how the state of a system is represented when its exact form is known only to lie within a set of possible states. In this case, the state of the system is *either* of the pair \{"hard"\}_m |\text{hard}\rangle_e, "soft"\}_m |\text{soft}\rangle_e\}, each with equal weight \(\frac{1}{2}\).

**But:** The result of a measurement (as given by the Projection Postulate and by our experience) is a *definite* outcome. In this case, the result is *either* |"hard"\}_m |\text{hard}\rangle_e or |"hard"\}_m |\text{hard}\rangle_e. It's *definitely* one of these two alternatives. It's not a weighted sum of them both!

**Aside:** This is the essential criticism of the decoherence approach to the measurement problem. A less severe criticism is the observation that the environment states \(|E_h\rangle\) and \(|E_s\rangle\) will in realistic situations never be *exactly* orthogonal to each other. The way in which the environment correlates itself to a pointer pointing to "hard" may "overlap" the way in which it correlates itself to a pointer pointing to "soft". Thus, in such realistic cases, when we "trace over the environment", we will *not* kill all the interference terms. However, one can show that the "off-diagonal" interference terms that survive in realistic cases have very short life-spans and go to zero very quickly. Of course this doesn't address the basic fact that, even when all interference terms are killed by decoherence, the state that results does not describe a definite outcome.

### III. Consistent Histories

**Def 1.** A *history* \(h\) is a time-sequence of facts, represented by time-indexed projection operators:

\[ h = \{P_1(t_1), P_2(t_2), ..., P_n(t_n)\} \]

**Example:** \(P_1(t_1)\) might be \(P_{|\text{hard}\rangle}\) which represents the property "The value of Hardness is hard". Or it might be \(P_{|a\rangle}\), which represents the property "The value of the property represented by A is a".

**Aside:** Projection operators evolve *via* the Schrödinger dynamics: \(P(t) = e^{iHt/H} P(0) e^{-iHt/H}\).
**Def 2.** The **probability** associated with a history \( h \) is given by:

\[
\Pr\_\psi(h) = \text{Tr}(P_n(t_n) \ldots P_2(t_2) P_1(t_1) P_{|\psi\rangle} P_1(t_1) P_2(t_2) \ldots P_n(t_n))
\]

where \( P_{|\psi\rangle} \) is the statistical operator associated with an initial state \( |\psi\rangle \).

**Motivation:** All the terms inside the trace commute with each other (one of the properties of the trace), so you can rearrange them into \( \text{Tr}(P_n(t_n) P_n(t_n) \ldots P_2(t_2) P_2(t_2) P_1(t_1) P_1(t_1)) \). And since projection operators are *idempotent*, this is equal to \( \text{Tr}(P_n(t_n) \ldots P_2(t_2) P_1(t_1) P_{|\psi\rangle}) \), which can be thought of as the Born Rule for the probability that the system, in the state \( |\psi\rangle \), has the "historical property" represented by the operator \( P_n(t_n) \ldots P_2(t_2) P_1(t_1) \).

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**Aside: A more detailed motivation for Def. 2:**

Consider the simple history \( h = \{ P_{|a\rangle} (t_1), P_{|\psi\rangle} (t_2) \} \). Suppose our system is in an initial state \( |\psi\rangle \).

**Now:** Calculate the *conditional probability* that it has the value \( b \) (of some operator \( B \)) at \( t_2 \), given it had the value \( a \) (of some other operator \( A \)) at time \( t_1 \). To do this, note that, if it possessed \( a \) at \( t_1 \), then at \( t_2 \) it will no longer be in the state \( |\psi\rangle \)! Measuring \( |\psi\rangle \) for \( a \) at \( t_1 \) changes \( |\psi\rangle \) to a new state, call it \( |\psi'\rangle \). \( |\psi'\rangle \) is the projection of \( |\psi\rangle \) onto the subspace spanned by \( |a\rangle \). It's (normalized) statistical operator is given by

\[
P_{|\psi'\rangle} = \frac{P_{|\psi\rangle} P_{|\psi\rangle} P_{|\psi\rangle}}{\text{Tr}(P_{|\psi\rangle} P_{|\psi\rangle} P_{|\psi\rangle})}
\]

**So:** \( \Pr\_\psi(\text{value of } B \text{ is } b \text{ in } |\psi'\rangle, \text{ given value of } A \text{ is } a \text{ in } |\psi\rangle) \)

\[
= \Pr\_\psi(\text{value of } B \text{ is } b \text{ in } |\psi'\rangle) \Pr\_\psi(\text{value of } A \text{ is } a \text{ in } |\psi\rangle)
\]

\[
= \text{Tr}(P_{|a\rangle} P_{|\psi\rangle}) \text{Tr}(P_{|a\rangle} P_{|\psi\rangle})
\]

\[
= \text{Tr} \left( \frac{P_{|\psi\rangle} P_{|\psi\rangle} P_{|\psi\rangle}}{\text{Tr}(P_{|\psi\rangle} P_{|\psi\rangle} P_{|\psi\rangle})} \right) \text{Tr}(P_{|a\rangle} P_{|\psi\rangle})
\]

\[
= \text{Tr} \left( \frac{P_{|\psi\rangle} P_{|\psi\rangle} P_{|\psi\rangle}}{\text{Tr}(P_{|\psi\rangle} P_{|\psi\rangle})} \right) \text{Tr}(P_{|a\rangle} P_{|\psi\rangle}) \quad \text{since } \text{Tr}(P_{|a\rangle} P_{|a\rangle} P_{|\psi\rangle}) = \text{Tr}(P_{|a\rangle} P_{|a\rangle}) \text{Tr}(P_{|\psi\rangle}) = \text{Tr}(P_{|a\rangle}) \text{Tr}(P_{|\psi\rangle})
\]

\[
= \text{Tr} \left( \frac{P_{|\psi\rangle} P_{|\psi\rangle} P_{|\psi\rangle}}{\text{Tr}(P_{|\psi\rangle} P_{|\psi\rangle})} \right) \text{Tr}(P_{|a\rangle} P_{|\psi\rangle}) \quad \text{since } \text{Tr}(\lambda A) = \lambda \text{Tr}(A), \text{ for any number } \lambda
\]

\[
= \text{Tr} \left( \frac{P_{|\psi\rangle} P_{|\psi\rangle} P_{|\psi\rangle}}{\text{Tr}(P_{|\psi\rangle} P_{|\psi\rangle})} \right) \frac{\text{Tr}(P_{|a\rangle} P_{|\psi\rangle})}{\text{Tr}(P_{|\psi\rangle} P_{|\psi\rangle})} \quad \text{since } \text{Tr}(P_{|a\rangle} P_{|a\rangle} P_{|\psi\rangle} P_{|\psi\rangle}) = \text{Tr}(P_{|a\rangle} P_{|a\rangle} P_{|\psi\rangle}) \text{Tr}(P_{|\psi\rangle}) = \text{Tr}(P_{|a\rangle} P_{|a\rangle} P_{|\psi\rangle}) \text{Tr}(P_{|\psi\rangle})
\]

Now extend this to histories of arbitrary length, and we get Def. 2.
**Def 3.** A family of histories is a time-sequence of sets of "exhaustive" facts. Each set of "exhaustive" facts is represented by a set of projection operators whose sum is the identity.

\[
\{\{P_1^{\alpha_1(t_1)}\}, \{P_2^{\alpha_2(t_2)}\}, ..., \{P_n^{\alpha_n(t_n)}\}\}, \quad \text{where each } \alpha_i = 1, ..., N \quad (N = \dim \text{ of Hilbert space})
\]

where set \(\{P_i^{\alpha_i(t_i)}\}\) consists of \(N\) projection operators, \(P_i^1(t_i), P_i^2(t_i), ..., P_i^N(t_i)\), such that \(P_i^1(t_i) + P_i^2(t_i) + ... + P_i^N(t_i) = I_N\).

**What this means:** The projection operators in any such set \(\{P_i^{\alpha_i(t_i)}\}\) represent all the possible values of the property associated with \(P_i(t_i)\).

**Note:** Histories can be embedded in families of histories:

\[
t = t_n \quad P_n^1(t_n) \quad P_n^2(t_n) \quad ... \quad ... \quad P_n^N(t_n) \quad \{P_n^{\alpha_n(t_n)}\}
\]

\[
t = t_2 \quad P_2^1(t_2) \quad P_2^2(t_2) \quad ... \quad ... \quad P_2^N(t_2) \quad \{P_2^{\alpha_2(t_2)}\}
\]

\[
t = t_1 \quad P_1^1(t_1) \quad P_1^2(t_1) \quad ... \quad ... \quad P_1^N(t_1) \quad \{P_1^{\alpha_1(t_1)}\}
\]

\[
t = 0 \quad P_{|\psi\rangle} \quad \{P_{|\psi\rangle}\}
\]

\(h = \{P_{|\psi\rangle}, P_1^1(t_1), P_2^2(t_2), ..., P_n^1(t_n)\}\)

\(h' = \{P_{|\psi\rangle}, P_1^2(t_1), P_2^N(t_2), ..., P_n^2(t_n)\}\)

\(h\) and \(h'\) are distinct histories within the family \(\{(P_{|\psi\rangle}, \{P_1^{\alpha_1(t_1)}\}, \{P_2^{\alpha_2(t_2)}\}, ..., \{P_n^{\alpha_n(t_n)}\}\}\)

**Note:** We can assign probabilities to histories within a family by means of Def 2.

**But:** Since this is based on the Born Rule, these will be quantum probabilities that exhibit "interference" effects.

**Are there histories within a given family that can be assigned classical probabilities?**
In other words, are there histories within a given family that do not "interfere" with each other? These would be histories $h$, $h'$ whose probabilities obey the classical Or-Addition Rule:

$$\Pr_q(h \text{ or } h') = \Pr_q(h) + \Pr_q(h').$$

**First:** Need a general expression for the disjunction, $h$ or $h'$, of two histories $h$, $h'$. Suppose $h_A$ and $h_B$ are histories that differ only in the property at $t = t_i$:

$$h_A = \{ P_1(t_1), ..., P_i^A(t_i), ..., P_n(t_n) \}$$
$$h_B = \{ P_1(t_1), ..., P_i^B(t_i), ..., P_n(t_n) \}$$

Now let the history, $h_A$ or $h_B$, be given by:

$$h_A \text{ or } h_B = \{ P_1(t_1), ..., P_i^A(t_i) + P_i^B(t_i), ..., P_n(t_n) \}$$

Then, for an initial state $|\psi\rangle$:

$$\Pr_q(h_A \text{ or } h_B) = \text{Tr}(P_n(t_n)...[P_i^A(t_i) + P_i^B(t_i)]...P_1(t_1)P_{|\psi\rangle}P_1(t_1)...[P_i^A(t_i) + P_i^B(t_i)]...P_n(t_n))$$

$$= \text{Tr}(P_n(t_n)...P_i^A(t_i)...P_1(t_1)P_{|\psi\rangle}P_1(t_1)...P_i^A(t_i)...P_n(t_n))$$
$$+ \text{Tr}(P_n(t_n)...P_i^B(t_i)...P_1(t_1)P_{|\psi\rangle}P_1(t_1)...P_i^B(t_i)...P_n(t_n))$$
$$+ \text{Tr}(P_n(t_n)...P_i^A(t_i)...P_1(t_1)P_{|\psi\rangle}P_1(t_1)...P_i^B(t_i)...P_n(t_n))$$
$$+ \text{Tr}(P_n(t_n)...P_i^B(t_i)...P_1(t_1)P_{|\psi\rangle}P_1(t_1)...P_i^A(t_i)...P_n(t_n))$$

$$= \Pr_q(h_A) + \Pr_q(h_B) + \text{interference terms}$$

**So:** The probabilities assigned to $h_A$ and $h_B$ by Def. 2 will be classical (i.e., obey the classical Or-Addition Rule) just when the interference terms vanish.

**Now:** Consider the general case:

$$h = \{ P_1(t_1), ..., P_n(t_n) \}$$
$$h' = \{ P_1'(t_1), ..., P_n'(t_n) \}$$

$$h \text{ or } h' = \{ [P_i(t_i) + P_i'(t_i)], ..., [P_i(t_i) + P_i'(t_i)], ..., [P_n(t_n) + P_n'(t_n)] \}$$

The probabilities assigned to $h$ and $h'$ by Def. 2 will be classical (i.e., obey the classical Or-Addition Rule) just when the general interference term vanishes:

$$\text{Tr}(P_n(t_n)...P_1(t_1)P_{|\psi\rangle}P_1'(t_1)...P_n'(t_n)) = 0$$
**Def. 4.** Two histories \( h = \{ P_1(t_1), ..., P_n(t_n) \} \), \( h' = \{ P_1'(t_1), ..., P_n'(t_n) \} \) are **consistent** just when \( \text{Tr}(P_n(t_n)...P_1(t_1)P_{\psi}P_1'(t_1)...P_n'(t_n)) = 0. \)

**Def. 5.** A **consistent family of histories** is a family of histories such that any two histories embeddable in it are consistent.

**What this means:** A consistent family of histories is a collection of histories that defines a classical sample space! You can assign classical probabilities to its members.

**Now add the notion of decoherence...**

**Def. 6.**
1. \( h \) is a **fine-grained history** just when all projection operators in \( h \) are 1-dim.
2. \( h' \) is a **coarse-graining of \( h \)** just when some projection operators in \( h' \) are sums of projection operators in \( h \).

**Can now say:**
- Fine-grained histories cannot in general be assigned classical probabilities.
- Course-grained histories can be assigned *approximate* classical probabilities, and these get better (more classical) as \( \text{Tr}(P_n(t_n)...P_1(t_1)P_{\psi}P_1'(t_1)...P_n'(t_n)) \to 0. \)
- As \( \text{Tr}(P_n(t_n)...P_1(t_1)P_{\psi}P_1'(t_1)...P_n'(t_n)) \to 0, \) such coarse-grained histories "decohere".

**What this is supposed to mean:**
Coarse-graining a family of histories corresponds to "tracing out the environment". The environment interacts with the coarse-grained histories to damp out the interference effects, rendering the family approximately consistent.

**Def. 7.** Two histories \( h = \{ P_1(t_1), ..., P_n(t_n) \} \), \( h' = \{ P_1'(t_1), ..., P_n'(t_n) \} \) are **decoherent** just when \( \text{Tr}(P_n(t_n)...P_1(t_1)P_{\psi}P_1'(t_1)...P_n'(t_n)) \to 0. \)

**Def. 5.** A **decoherent family of histories** is a family of histories such that any two histories embeddable in it are decoherent.
Characteristics of the Consistent/Decoherent Histories (CH) Approach

- Replaces states of a physical system with histories a physical system.
- The properties (projection operators) that make up a history evolve only via the Schrödinger dynamics (no Projection Postulate).
- Identifies a way to associate a probability with a history (Def. 2).
- Identifies a condition that picks out those families of histories that are classical (or approximately classical) (Defs. 4, 5).

Problems:

1. How are alternative histories within a decoherent family to be interpreted?
   - Is one such history our actual history and the others just possible? This is our experience, but CH is silent as to how to specify the actual history from possible alternatives.
   - Are all histories within a decoherent family occurent? If so, then how are probabilities explained? (This is the Problem of Probabilities that Many Worlds faces.)

2. How are alternative decoherent families to be interpreted?
   Any history \( h \) can be embedded in many different mutually incompatible decoherent families (any one of which defines an approximately classical probability space). Which do we choose in order to calculate the probability of \( h \)? (This is the Preferred Basis Problem that Many Worlds faces.)

Problems 1 & 2 Combined:

Seem to indicate that CH isn't fundamentally different from Many Worlds. All CH does is replace world-talk with history-talk, and adds a criterion for identifying histories that behave "classically".

3. General Problem with the Notion of Decoherence

"Tracing over the environment" (or "coarse-graining" histories) does not pick out a unique measurement/interaction outcome. It does not effect a "collapse" of superposed states (or "interfering" histories). So it cannot be appealed to in order to reconcile superpositions (or "interfering" histories) with our experience of unique outcomes.