12. Modal Interpretations and Quantum Logic

I. Modal Interpretations

**Motivation:** Let’s return to using Hilbert spaces to represent QM state spaces, and operators to represent properties. The *Kochen-Specker Theorem* says that the properties associated with a given Hilbert space $\mathcal{H}$ cannot *all* have values at the same time (for $\dim\mathcal{H} \geq 3$).

**One Option for Avoiding KS:** Claim that, for any Hilbert space $\mathcal{H}$, *some* (not all) properties defined on $\mathcal{H}$ always have determinate values (even in superpositions), others do not.

**Recall Bohm:** One property (position) is always determinate (always has a value). All other properties are “contextual” — their values depend on how they are measured. Technically, this is sufficient to avoid the KS Theorem (recall, for instance, that *non-contextuality* was one of its assumptions).

**Modal Interpretations Claim:**

> For any Hilbert space $\mathcal{H}$, at any given time, there is a subset of properties that are always determinate (always possess values). The QM probabilities for these properties are epistemic: for these properties, probabilities represent our ignorance of their actual values.

“Modal” means “possible”. The always-determinate properties are those properties that *any* state (even superposed states) can possibly possess.

**Important: Modal Interpretations reject the Eigenvector/Eigenvalue Rule.**

The *Eigenvector/Eigenvalue Rule* says:

\[
\begin{align*}
\left( \text{A state possesses the value } \lambda \text{ of a property represented by an operator } O \right) & \iff \text{“if and only if”} \left( \text{that state is an eigenvector of } O \text{ with eigenvalue } \lambda \right) \\
\end{align*}
\]

Modal interpretations agree with the “if” part ($\iff$), but not the “only if” ($\Rightarrow$). They allow for states to possess values of properties, even when the states are not eigenvectors of the relevant operators! So they say: For any given state $|Q\rangle$, in addition to those properties for which $|Q\rangle$ is an eigenvector, there are *other* properties for which $|Q\rangle$ also possesses values.

**Initial task for Modal Interpretations:** Identify the subset of determinate properties.

*Modal interpretations can differ on how they identify this subset.*

At any given time, the subset of determinate properties for a given physical system is given by the basis states of a certain expansion of the system’s state vector. Namely, the biorthogonal expansion of the system’s state vector.

**Biorthogonal Decomposition Theorem:** Let $|Q\rangle$ be a vector in the product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then there is a basis $|a_1\rangle, |a_2\rangle, \ldots, |a_N\rangle$ of $\mathcal{H}_1$, and a basis $|b_1\rangle, |b_2\rangle, \ldots, |b_N\rangle$ of $\mathcal{H}_2$ such that $|Q\rangle$ can be expanded as:

$$|Q\rangle = c_{11}|a_1, b_1\rangle + c_{22}|a_2, b_2\rangle + \ldots + c_{NN}|a_N, b_N\rangle$$

And, if $|c_{11}| \neq |c_{22}| \neq \ldots \neq |c_{NN}|$, then these bases are unique.

**Recall:** In general, if $|g\rangle$ and $|h\rangle$ are bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, then any vector $|Q\rangle$ can be expanded as $|Q\rangle = c_{11}|g_1, h_1\rangle + c_{12}|g_1, h_2\rangle + \ldots + c_{21}|g_2, h_1\rangle + c_{22}|g_2, h_2\rangle + \ldots$. The Biorthog Decomp Theorem says that there are some bases in which the “cross terms” with coefficients $c_{12}, c_{21}, \text{etc}$, all vanish. And these bases will be unique just when all the remaining coefficients are different from each other.

**Why do we want bases in which the “cross terms” vanish?** Because these are the bases we (would like to) associate with post-measurement systems. Consider a composite system composed of a measuring device $m$ and a black electron $e$. After measurement, we’d like to say the system is in the state:

$$|Q\rangle = a|\text{“hard”}_m, \text{“hard”}_e\rangle + b|\text{“soft”}_m, \text{“soft”}_e\rangle$$

This is in the form of a biorthogonal expansion! To see this, note that in general $|Q\rangle$ should be expanded as:

$$|Q\rangle = a|\text{“hard”}_m, \text{“hard”}_e\rangle + c|\text{“hard”}_m, \text{“soft”}_e\rangle + d|\text{“soft”}_m, \text{“hard”}_e\rangle + b|\text{“soft”}_m, \text{“soft”}_e\rangle$$

since $|\text{“hard”}_m\rangle, |\text{“soft”}_m\rangle$ is a basis for the state space of $m$, and $|\text{“hard”}_e\rangle, |\text{“soft”}_e\rangle$ is a basis for the state space of $e$. Now if these bases are biorthogonal bases, then $c = d = 0$, and we get our original result. So if we just stipulate that properties associated with biorthogonal expansion bases are always determinate, then perhaps we can avoid having to use the Projection Postulate at the end of a measurement. And this is what KHD in fact says:

**KHD Rules (replacements for Projection Postulate):**

1. **Rule 1:** For any physical system $S$ that is composed of two subsystems $S_1$ and $S_2$, there are some properties for which $S$ always possesses values. To identify them:
   i. Expand the state vector $|Q\rangle$ for $S$ in its biorthogonal decomposition.
   ii. The biorthogonal basis states $|a_1\rangle, |a_2\rangle, \ldots, |a_N\rangle$, and $|b_1\rangle, |b_2\rangle, \ldots, |b_N\rangle$ are the eigenvectors of the determinate properties, call them $A$ and $B$.
   iii. The subsystems $S_1$ and $S_2$ can be said to have determinate values for the properties $A$ and $B$, so identified.

2. **Rule 2:** If $S$ is in the state $|Q\rangle$, then the probability that $S_1$ has the value $a_i$ of the property $A$ is $c_{ii}^2$, and the probability that $S_2$ has the value $b_i$ of the property $B$ is $c_{ii}^2$ (Born Rule).
**Why this is helpful:**

Suppose a system is in a state represented by \(|Q\rangle = a|\text{“hard”}\rangle_m|\text{hard}\rangle_e + b|\text{“soft”}\rangle_m|\text{soft}\rangle_e\) (and suppose \(a \neq b\)).

**The Standard Interpretation says:**

“This is a state in which \(e\) can’t be said to have the Hardness property, and \(m\) can’t be said to be indicating “hard” or “soft”.”

**KHD say:**

“This is a state in which \(e\) does have a definite value of Hardness, and \(m\) is definitely pointing to either “hard” or “soft” (even though we don’t know what Hardness value \(e\) has, and we don’t know where \(m\) is pointing).”

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**Essential Characteristics of Modal Interpretations**

(A) Rejection of Eigenvector/Eigenvalue Rule

(B) Rejection of Projection Postulate

(C) Probabilities are epistemic

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**3 Problems with KHD**

1. **Non-uniqueness of biorthogonal expansions.**
   - Consider the biorthogonal expansion \(|Q\rangle = c_{11}|a_1\rangle|b_1\rangle + c_{22}|a_2\rangle|b_2\rangle + \ldots + c_{NN}|a_N\rangle|b_N\rangle\).
   - The Biorthog Decomp Theorem says this expansion is unique, provided that \(|c_{11}| \neq |c_{22}| \neq \ldots \neq |c_{NN}|\).
   - If this does not hold; i.e., if any of the expansion coefficients are equal, then there will be other biorthogonal expansions of \(|Q\rangle\), in fact infinitely many.

   **Ex:**
   - \(|Q\rangle = \sqrt{\frac{1}{3}}|\text{“hard”}\rangle_m|\text{hard}\rangle_e + \sqrt{\frac{1}{3}}|\text{“soft”}\rangle_m|\text{soft}\rangle_e\)
     - \(= \sqrt{\frac{1}{3}}|\text{“black”}\rangle_m|\text{black}\rangle_e + \sqrt{\frac{1}{3}}|\text{“white”}\rangle_m|\text{white}\rangle_e\)
     - \(= \ldots\)
   - In such cases, KHD Rule 1 will say that infinitely many properties will have definite values at any given time, and this violates the KS Theorem!

**ASIDE 1.** Strictly speaking, KS doesn’t apply to 2-dim Hilbert spaces. So it’s okay to claim that all properties associated with the 2-dim subsystems in this example have simultaneous values. But the non-uniqueness claim holds for more complex composite systems whose subsystems have greater than 2 dimensions.

**ASIDE 2.** There are more general versions of modal interpretations that avoid this problem. It turns out that another type of state expansion is unique: what’s called the spectral decomposition of the density operator associated with the state of a system. Density operators are generalized projection operators. Any state vector \(|Q\rangle\) can be associated with a particular projection operator \(P_{|Q\rangle}\), which projects onto the ray defined by \(|Q\rangle\). A density operator \(W\) is, in general, a (convex) sum of projection operators: \(W = a_1P_1 + a_2P_2 + \ldots\), where \(a_1 + a_2 + \ldots = 1\). It represents a weighted sum of state vectors (each associated with a given 1-dim projection operator). So it represents the state of a physical system whose exact state is not known, but falls within a range of known states. And it turns out that any \(W\) can be uniquely decomposed into such a sum (its spectral decomposition). The Spectral Modal Interpretation essentially says: The determinate properties of a system are given by the operators associated with the terms in the spectral decomposition of the system’s density operator.
2. **Dynamics for determinate properties.**

- All modal interpretations (not just KHD) say that, at any given time, a physical system possesses the values of some subset of properties (in addition to, and including those given by the EE Rule).
- Let $\text{Det}_t$ be this set of determinate properties at time $t$. This set can change from moment to moment! In other words, $\text{Det}_t$ may be different from $\text{Det}_{t'}$ for $t \neq t'$.
- **Example:** In the KHD version, $\text{Det}_t$ depends on the component states of the composite system, and these component states may change over time.
- So all modal interpretations need to tell us how $\text{Det}_t$ changes over time. They need to give us a *dynamics* for the determinate properties. KHD does not specify this.

\[
\text{Aside: Bohmian Mechanics can be considered as a modal interpretation in which the property dynamics (for the position property) is given by Bohm's Equation.}
\]

3. **Imperfect Measurements.**

**Claim:** The biorthogonal expansions of post-measurement states that KHD identifies represent *ideal* perfect measurements. For *actual* imperfect measurements, KHD does not pick out the right post-measurement properties.

**Recall:** KHD seemed to work for the post-measurement state:

\[
|Q\rangle = a|\text{"hard"}_m\rangle_{\text{m}}|\text{hard}\rangle_{\text{e}} + b|\text{"soft"}_m\rangle_{\text{m}}|\text{soft}\rangle_{\text{e}} \quad \text{(suppose } a \neq b)\]

This is in the form of a biorthogonal expansion, so KHD says: the electron has a definite value of *Hardness*.

**BUT:** In reality, the post-measurement state will *really* be:

\[
|J\rangle = c|\text{"hard"}_m\rangle_{\text{m}}|\text{hard}\rangle_{\text{e}} + d|\text{"soft"}_m\rangle_{\text{m}}|\text{soft}\rangle_{\text{e}} + f|\text{"hard"}_m\rangle_{\text{m}}|\text{soft}\rangle_{\text{e}} + g|\text{"soft"}_m\rangle_{\text{m}}|\text{hard}\rangle_{\text{e}}
\]

**Error terms!** Represent the fact that real measuring devices will never perfectly correlate pointers with *Hardness* property. For realistic measuring devices, $f$ and $g$ can be made very small, but they will never vanish.

**NOTE:** $|J\rangle$ does have a biorthogonal expansion (guaranteed by the Biorhong Decomp Theorem), but it will *not* be the one that KHD needs:

\[
|J\rangle = k|\text{w}\rangle_{\text{m}}|\text{grump}\rangle_{\text{e}} + l|\text{w}\rangle_{\text{m}}|\text{gromp}\rangle_{\text{e}}
\]

*Grump* and *gromp* are values of some property (they are eigenvectors of some operator), but definitely not *Hardness*. So if we were to use KHD Rule 1 on $|J\rangle$, we’d have to say that, after a *Hardness* measurement, the electron is either *grump* or *gromp*. We couldn’t say it is either *hard* or *soft*. We couldn’t say that it has a definite *Hardness* value.
II. Quantum Logic Interpretation

Motivation: When a physical system is in a state represented by a superposition, we can’t use classical logic to describe the properties it possesses. Consider the state \( |Q\rangle = a|\text{“hard”}\rangle_m|\text{hard}\rangle_e + b|\text{“soft”}\rangle_m|\text{soft}\rangle_e \). Under the standard interpretation, an electron in this state:

(a) Can’t be said to be hard.
(b) Can’t be said to be soft.
(c) Can’t be said to be both hard and soft.
(d) Can’t be said to be neither hard nor soft.

Perhaps to make sense of such superposed states, we need to change our logic!

Goal: To develop a “quantum” logic that will allow us to say meaningful things about the properties of states in superpositions.

General Idea: Classical mechanics (CM) represents properties in a certain way (as functions on a phase space), and it turns out that this way has a structure associated with it that can be identified with the structure of classical logic. We know that QM represents properties in a different way (as operators on a Hilbert space). So the structure of QM properties is different from that of CM properties, and hence, classical logic. Our new quantum logic will be based on the structure associated with operators on a Hilbert space.

The Logic of Classical Mechanics

Recall: CM state space phase space (set of points)
CM states points
CM properties functions

NOW: Consider the property "The value of property A is a". Recall in QM, this property can be represented by a projection operator. In CM, this property is represented by a subset of phase space; namely, the collection of all phase space points that represent states in which the value of property A is a.
Phase space $\Omega$ and 3 subsets: $P$, $Q$, $R$. $P$ might represent the property "The value of property $A$ is $a$". This would mean that all points in $P$ represent states in which the value of property $A$ is $a$. $Q$ might represent the property "The value of property $B$ is $b$". The intersection $P \cap Q$ would then represent the property "The value of property $A$ is $a$ and the value of property $B$ is $b$".

Note that all properties that a physical system can possess can be phrased in terms of properties of this sort (of the sort "The value of property $X$ is $x". So subsets of points represent $CM$ properties in general. It now turns out that there is a correspondence between operations on sets and classical logic connectives:

<table>
<thead>
<tr>
<th>set operation</th>
<th>classical logic connectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cap$ (intersection)</td>
<td>$\cdot_c$ (and)</td>
</tr>
<tr>
<td>$\cup$ (union)</td>
<td>$\lor_c$ (or)</td>
</tr>
<tr>
<td>$\neg$ (complement)</td>
<td>$\sim_c$ (not)</td>
</tr>
</tbody>
</table>

**To see this:**

i. Let the set $P$ represent the sentence $p = "The value of property $A$ is $a."$
ii. Let the set $Q$ represent the sentence $q = "The value of property $B$ is $b."$
iii. Let the set $R$ represent the sentence $r = "The value of property $C$ is $c."$

**THEN:**

i. $P \cap Q$ represents "The value of property $A$ is $a$ and the value of property $B$ is $b."$
   Which is just $p \cdot_c q$!
ii. $P \cup R$ represents "The value of property $A$ is $a$ or the value of property $C$ is $c."$
   Which is just $p \lor_c q$!
iii. $\neg P$ represents "The value of property $A$ is not $a"$ Which is just $\sim_c p$!

**ASIDE:** From a mathematical point of view, the reason there is a correspondence between set operations and classical logic connectives is that both are concrete representations of a type of abstract algebra: what's called a **Boolean algebra**. In other words, $CM$ properties, sets, and classical logic all have the same general (Boolean algebraic) structure.
NOTE: We could construct classical logic just as the logic of the structure of CM properties. (We could define all the classical logic connectives in terms of their set-theoretic counterparts.) This is an empirical approach to logic! Why do we use classical logic to describe the world? Because of the way classical physics describes the world. This suggests that, when the physics changes, so should the logic!

The Logic of Quantum Mechanics

Recall: QM state space Hilbert space (vector space)
QM states vectors
QM properties operators

NOW: Consider the property "The value of property A is a". In QM, this property can be represented by a projection operator \( P_{|a\rangle} \), which projects any vector onto the ray defined by the vector \(|a\rangle\) (the eigenvector of A with eigenvalue \(a\)). (See the lecture on the KS Theorem for a reminder on projection operators.) A ray (i.e., a line) in a Hilbert space \(\mathcal{H}\) is just a 1-dimensional subspace of \(\mathcal{H}\). So for QM, properties of the sort "The value of property X is x" are represented by subspaces (and not subsets).

Subspaces are related by 3 basic operations:

<table>
<thead>
<tr>
<th>Subspace operations</th>
<th>V (\cap) W = {all vectors in both V and W}</th>
<th>V (\oplus) W = {all linear combinations of vectors from V and W}</th>
<th>V(\perp) = {all vectors that are orthogonal to vectors in V}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\cap) (intersection)</td>
<td>(\oplus) (linear span)</td>
<td>(\perp) (orthocomplement)</td>
<td></td>
</tr>
</tbody>
</table>

Note: The linear span lets you construct a subspace of a higher dimension. If V and W are both 1-dim subspaces (rays), then their linear span \(V \oplus W\) is a 2-dim subspace -- a plane containing all vectors of the form \(a|v\rangle + b|w\rangle\), where \(|v\rangle\) is in V and \(|w\rangle\) is in W. In general, these higher dimensional subspaces also correspond to projection operators. \(V \oplus W\) corresponds to the projection operator \(P_{V \oplus W} = P_{|v\rangle} + P_{|w\rangle}\). (You can check that \(P_{V \oplus W}\) has the 2 essential characteristics of projection operators.)

ASIDE: A subspace of a Hilbert space \(\mathcal{H}\) is a subset of \(\mathcal{H}\) closed under vector addition and scalar multiplication (in other words, a subspace is just a part of \(\mathcal{H}\) that is itself a vector space). There is a 1-1 correspondence between projection operators and subspaces.

ASIDE: Why these 3 operations? It turns out linear subspaces are closed under these operations: The result of applying any of them is another subspace. This is why the set operation of union \(\cup\) is not included: The union of 2 subspaces is not, in general, another subspace. (Suppose V and W are 1-dim subspaces. Then \(V \cup W\) is the set of all vectors in both V and W; i.e., all vectors lying in the rays defined by V and W. This set is not a subspace: it’s not closed under vector addition, for instance. The sum of two vectors from V and W may not itself be in \(V \cup W\).)
**SO:** The structure of QM properties is given by the **subspace structure** of a Hilbert space (as opposed to the **subset structure** of a phase space). One important property of the subspace structure: **It is not distributive:**

\[
X \cap (V \oplus W) = (X \cap V) \oplus (X \cap W).
\]

**Proof:** Suppose \( V, W \) and \( X \) are subspaces of \( \mathcal{H} \) and suppose \( X \) is a subset of \( V \oplus W \). (In other words, any vector \( |x⟩ \) in \( X \) can be written as \( |x⟩ = a|v⟩ + b|w⟩ \), with \( |v⟩ \) in \( V \) and \( |w⟩ \) in \( W \).) Then \( X \cap (V \oplus W) = X \). But \( (X \cap V) \oplus (X \cap W) = 0 \oplus 0 = 0 \). (Here 0 is the “zero” subspace that contains no vectors.)

Since Boolean algebras are distributive, this means that the subspace structure of QM properties is not a Boolean algebra; so it really is different from the subset structure of CM properties and the structure of classical logic! Let’s now try to define a new quantum logic in terms of the following correspondence:

<table>
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<th>subspace operation</th>
<th>quantum logic connectives</th>
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</thead>
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</tr>
<tr>
<td>( \perp ) (orthocomplement)</td>
<td>( \sim_q ) (not)</td>
</tr>
</tbody>
</table>

The things that quantum logic connectives act on are sentences (like classical logic) of the type "The value of property \( A \) is \( a \)." Now think of these sentences as represented by subspaces (instead of subsets). In CM, to say "The value of property \( A \) is \( a \)" is to say "The state of the system lies in the subset \( P \)." In QM, to say "The value of property \( A \) is \( a \)" is to say "The state of the system lies in the subspace \( V \)." 

**Now:** Let the subspace \( V \) represent the sentence \( v = "\text{The value of property } A \text{ is } a_i."\) Let the subspace \( W \) represent the sentence \( w = "\text{The value of property } B \text{ is } b_i."\) Let the subspace \( X \) represent the sentence \( x = "\text{The value of property } C \text{ is } c_i."\)

**Then we stipulate:**

i. The QL sentence \( v \bullet_q w \) represents "The value of property \( A \) is \( a \) and the value of property \( B \) is \( b \), which means "The state of the system lies in the subspace \( V \cap W \)."

ii. The QL sentence \( v \lor_q w \) represents "The value of property \( A \) is \( a \) or the value of property \( C \) is \( c \), which means "The state of the system lies in the subspace \( V \oplus W \)."

iii. The QL sentence \( \sim_q v \) represents "The value of property \( A \) is not \( a \), which means "The state of the system lies in the subspace \( V^\perp \)."
Why this is supposed to help: We can now claim that, as a matter of QL (Quantum Logic):

1. “A has a definite value” is a QL tautology (always a true statement), for all properties A.

Why?

\[
\begin{align*}
\text{"A has a} & \quad \text{means} & \quad \text{"The value of property A is } a_1, \text{ or} \\
\text{definite value"} & \quad \text{the value of property A is } a_2, \text{ or } ... \text{ or} \\
& & \text{the value of property A is } a_N. \quad \text{means} & \quad \text{"The state of the system} \\
& & \text{lies in } V_1 \oplus V_2 \oplus ... \oplus V_N. \\
\end{align*}
\]

This last statement is always true, since \( V_1 \oplus V_2 \oplus ... \oplus V_N = \mathcal{H} \)

Here \( V_1, V_2, ..., V_N \) are the subspaces spanned by the eigenvectors \( |a_1\rangle, |a_2\rangle, ..., |a_N\rangle \) of A. These eigenvectors form a basis for \( \mathcal{H} \), so the span of their subspaces is just \( \mathcal{H} \) itself! And it’s always true that the state of the system will always be somewhere in \( \mathcal{H} \).

SO: As a matter of QL, all properties always have definite values at all times -- even properties of measuring devices in superposed states!

2. Statements about incompatible properties possessing simultaneous values are contradictory (so they are meaningless).

Example: “Electron e is both soft and black” is a statement about incompatible properties (Color and Hardness) possessing simultaneous values. And QL says this statement has no meaning.

Why?

Suppose \( A \) and \( B \) are properties with non-orthogonal eigenvectors (\( i.e., they are incompatible properties like Hardness and Color \)). Then:

\[
\begin{align*}
\text{"Property A has a} & \quad \text{means} & \quad \text{"(The value of A is } a_1 \text{ and the value} \\
\text{value and property} & \quad \text{of B is } b_1 \text{) or (the value of A is } a_1 \\
B \text{ has a} & \quad \text{and the value of B is } b_2 \text{) or ... or} \\
\text{value"} & \quad \text{the value of A is } a_2 \text{ and the value} \\
& & \text{of B is } b_1 \text{) or ... or (the value of A} \\
& & \text{is } a_N \text{ and the value of B is } b_N. \quad \text{means} & \quad \text{"The state of the system lies in} \\
& & \text{"The state of the system lies in} \\
& & (V_1 \cap W_1) \oplus (V_1 \cap W_2) \oplus ... \oplus \\
& & (V_2 \cap W_1) \oplus (V_2 \cap W_2) \oplus ... \oplus \\
& & (V_N \cap W_N)" \\
& & (V_N \cap W_N)" \\
\end{align*}
\]

This last statement is a QL contradiction (it’s always false). We conclude that it is (quantum) logically contradictory to say that a system possesses simultaneous values for properties \( A \) and \( B \).

Here \( V_1, ..., V_N \) and \( W_1, ..., W_N \) are the subspaces spanned by the eigenvectors of \( A \) and \( B \), respectively. Since \( V_i \) and \( W_j \) are disjoint for any \( i, j \) (they have no vectors in common), all the intersection terms vanish, and we’re left with \( 0 \oplus 0 \oplus ... \oplus 0 = 0 \), where \( 0 \) is the zero subspace (contains no vectors). But the state of the system should be somewhere. So it’s always false that it’s “nowhere”.


**Major Problem**

If QL says all properties of a system have definite values at all times, this gets around the *Measurement Problem*, but it then runs up against the *Kochen-Specker Theorem*!

**One option:**

To say that every property always has a value is not to say that there is always a value that every property has.

**Example:**

- Let $V_1, V_2, \ldots, V_N$ and $W_1, W_2, \ldots, W_N$ be the 1-dim subspaces spanned by the eigenvectors $|a_1\rangle, |a_2\rangle, \ldots, |a_N\rangle$ and $|b_1\rangle, |b_2\rangle, \ldots, |b_N\rangle$ of two operators $A, B$.
- Then $W_i \cap (V_1 \oplus V_2 \oplus \ldots \oplus V_N)$ represents the sentence:
  
  "The value of property $B$ is $b_i$ and property $A$ has a definite value."  
  
  (1)

- And $(W_i \cap V_1) \oplus (W_i \cap V_2) \oplus \ldots \oplus (W_i \cap V_N)$ represents the sentence:
  
  "(The value of $B$ is $b_i$ and the value of $A$ is $a_1$) or (the value of $B$ is $b_i$ and the value of $A$ is $a_2$) or ... or (the value of $B$ is $b_i$ and the value of $A$ is $a_N$)."  
  
  (**)

  which means  "The value of $B$ is $b_i$ and the value of $A$ lies in $\{a_1, a_2, \ldots, a_N\}$."  

  which means  "The value of $B$ is $b_i$ and there is a value that $A$ has."  

- Now $W_i \cap (V_1 \oplus V_2 \oplus \ldots \oplus V_N) \neq (W_i \cap V_1) \oplus (W_i \cap V_2) \oplus \ldots \oplus (W_i \cap V_N)$.
- So the sentences (1) and (**) do not mean the same thing!
- Thus to say that property $A$ has a definite value is not to say that there is some definite value $(a_1, a_2, \ldots, a_N)$ it has!

Properties like $A$ above are called "*disjunctive properties*". They have a definite value, which is the *disjunction* $(a_1 \lor a_2 \lor a_3 \lor \ldots)$, but there is no definite value $a_1$ individually, or $a_2$ individually, etc., that they have!

**Lingering Problem:** QL (under this view) is motivated by the desire to view properties realistically (i.e., to be able to say that properties do have definite values at all times). Does the notion of a disjunctive property really provide us with an adequate notion of property realism?