

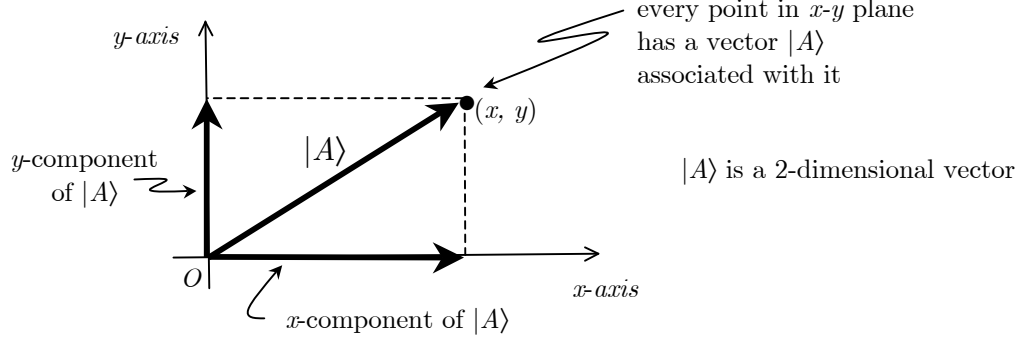
03. Vectors, Vector Spaces, and Operators (Albert Chap 2)

I. Vectors and Vector Spaces

1. Vectors

a vector = a magnitude (“length”) *and* a direction ← a number (or “scalar”) is just a magnitude

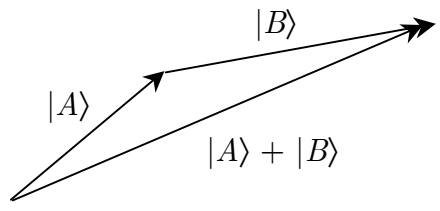
One way to represent vectors:



In this example, the collection of all vectors $|A\rangle, |B\rangle, |C\rangle, \dots$ forms a 2-dimensional *vector space*, call it V .

2. Vector addition

To add vectors $|A\rangle$ and $|B\rangle$ in V , place tail of $|B\rangle$ to head of $|A\rangle$ to form a third vector in V , $|A\rangle + |B\rangle$, whose tail is the tail of $|A\rangle$ and whose head is the head of $|B\rangle$:



3. Scalar (number) multiplication

Numbers can be multiplied to vectors. The result is another vector.

ex: $5|A\rangle$ is a vector in V that you get by multiplying the vector $|A\rangle$ by the number 5.

Numbers can be either *real* (like 5) or *complex* (like $5 + i6$). If you allow complex scalar multiplication, then you get a *complex vector space*.

i is the unit imaginary number defined by $i = \sqrt{-1}$

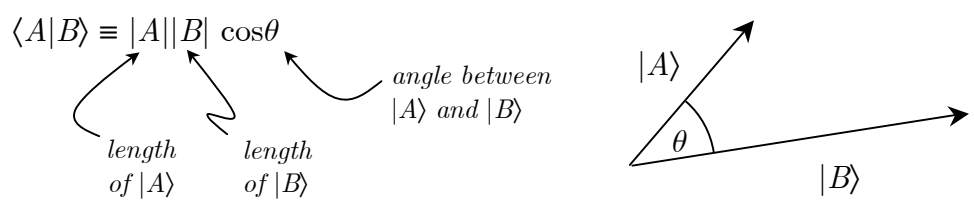
ASIDE: Technically, what's called a (real or complex) linear vector space V consists of the following:

- (a) A set Ω of objects $|A\rangle, |B\rangle, |C\rangle, \dots$ called vectors.
- (b) A map $+$ (*vector addition*) that assigns to any two vectors $|A\rangle, |B\rangle$ in Ω another vector $|A\rangle + |B\rangle$ in Ω .
- (c) A map $*$ (*scalar multiplication*) that assigns to any vector $|A\rangle$ in Ω and any number n , another vector $n*|A\rangle$ in Ω .
- (d) The following rules for vector addition: For any vectors $|A\rangle, |B\rangle, |C\rangle$ in Ω ,
 - (i) $|A\rangle + |B\rangle = |B\rangle + |A\rangle$ (*commutativity*)
 - (ii) $(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle)$ (*associativity*)
 - (iii) There is an identity vector $|I\rangle$ such that $|A\rangle + |I\rangle = |A\rangle$. (*identity*)
 - (iv) Every vector $|A\rangle$ has an inverse $-|A\rangle$ such that $|A\rangle + (-|A\rangle) = |I\rangle$. (*inverse*)
- (e) The following rules for scalar multiplication: For any vectors $|A\rangle, |B\rangle$ in Ω and any numbers n, m ,
 - (i) $n*(m*|A\rangle) = (nm)*|A\rangle$
 - (ii) $(n + m)*|A\rangle = n*|A\rangle + m*|A\rangle$
 - (iii) $n*(|A\rangle + |B\rangle) = n*|A\rangle + n*|B\rangle$
 - (iv) $1*|A\rangle = |A\rangle$ (where 1 is the (real or complex) number identity)

Note: The star symbol $*$ for scalar multiplication is normally dropped for notational convenience.

4. Inner (or "dot") -product

Vectors can be multiplied to each other. One type of vector multiplication is called the inner-product. The inner-product of two vectors $|A\rangle, |B\rangle$ is written as $\langle A|B\rangle$ and is a *number* defined by:



Note: This means $\langle A|A\rangle = |A| |A| \cos(0) = |A|^2$

So the length of a vector $|A\rangle$ is given by $|A| = \sqrt{\langle A|A\rangle}$

ASIDE: A linear vector space V equipped with an inner-product is called an inner-product space. The inner-product can be defined as a map that assigns to any two vectors $|A\rangle, |B\rangle$ in V , a number, and that satisfies the following rules:

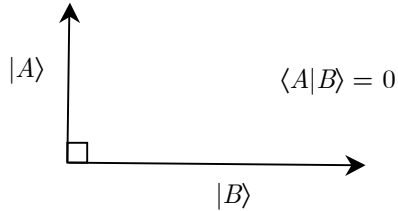
- (i) $\langle A|B\rangle = \langle B|A\rangle^*$ (*symmetry*)
- (ii) $\langle A|A\rangle \geq 0$ (*positive-definiteness*)
- (iii) $\langle A|(n*|B\rangle + m*|C\rangle) = n\langle A|B\rangle + m\langle A|C\rangle$ (*linearity*)

This star is complex conjugation: When the number n is complex, and hence can be written $n = a + ib$, for real a, b , then $n^* = a - ib$. When the number n is real, then $n^* = n$.

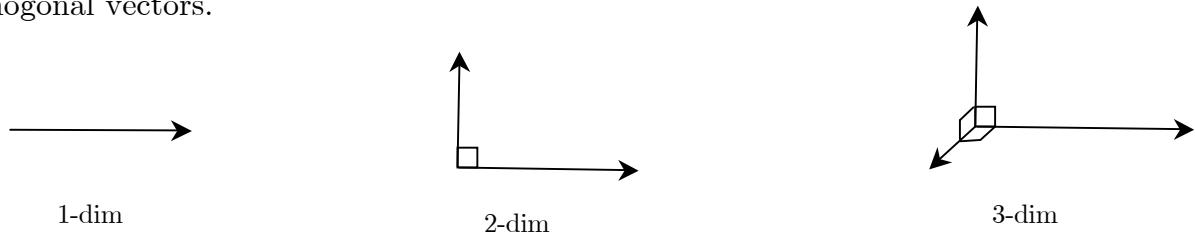
The star down here is scalar multiplication. Again, we normally drop the star symbol here for convenience.

5. Two non-zero-length vectors $|A\rangle$ and $|B\rangle$ are orthogonal (perpendicular) just when their inner-product is zero: $\langle A|B\rangle = 0$.

Check: $\langle A|B\rangle = |A| |B| \cos\theta$
 $= 0$ just when $\theta = 90^\circ$, when $|A| \neq 0 \neq |B|$



6. The dimension of a linear vector space is equal to the maximum number N of mutually orthogonal vectors.

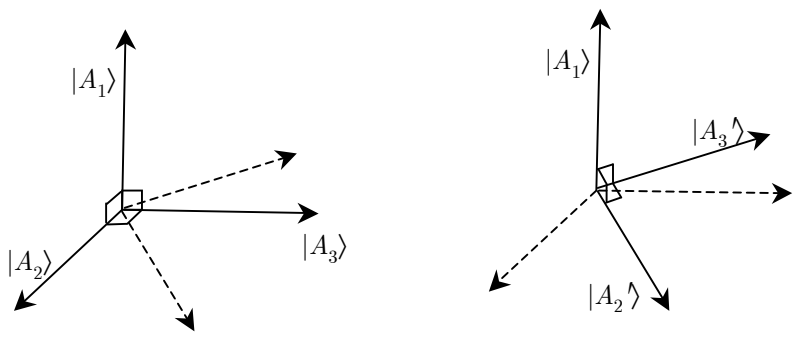


7. An orthonormal basis of an N -dimensional vector space is a set of N mutually orthogonal vectors, each with unit length (or “norm”).

Important Note: Any N -dim vector space can have many different orthonormal bases!

ex: Let $N = 3$. Then here are two different sets of orthonormal bases:

$$|A_1\rangle, |A_2\rangle, |A_3\rangle \quad \text{and} \quad |A_1'\rangle, |A_2'\rangle, |A_3'\rangle$$



8. In an N -dimensional vector space, any vector $|B\rangle$ can be expanded in terms of any orthonormal basis:

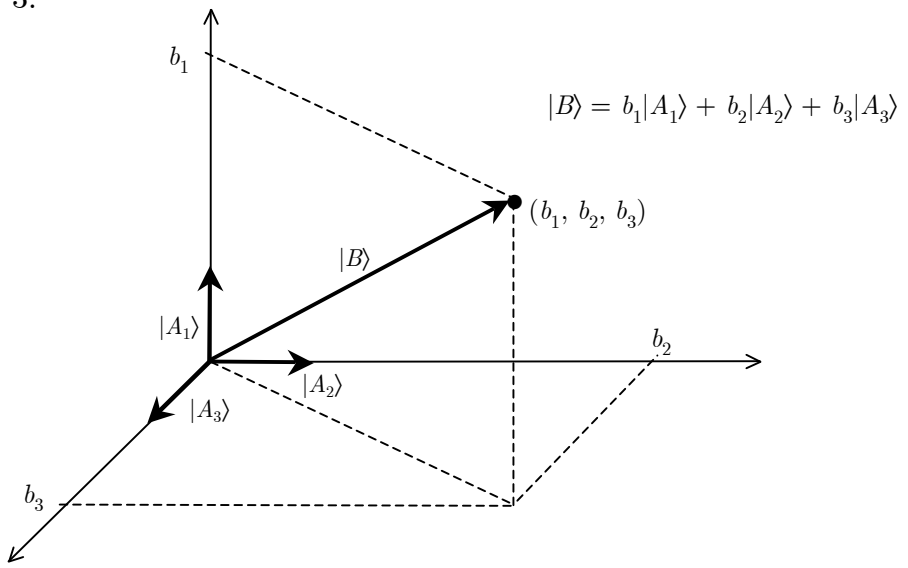
$$|B\rangle = b_1|A_1\rangle + b_2|A_2\rangle + \dots + b_N|A_N\rangle$$

where $b_i = \langle A_i|B\rangle$

these are numbers called "expansion coefficients"

Check: $\langle A_i|B\rangle = \langle A_i|(b_1|A_1\rangle + b_2|A_2\rangle + \dots + b_i|A_i\rangle + \dots + b_N|A_N\rangle)$
 $= \langle A_i|b_1|A_1\rangle + \langle A_i|b_2|A_2\rangle + \dots + \langle A_i|b_i|A_i\rangle + \dots + \langle A_i|b_N|A_N\rangle$
 $= b_1\langle A_i|A_1\rangle + b_2\langle A_i|A_2\rangle + \dots + b_i\langle A_i|A_i\rangle + \dots + b_N\langle A_i|A_N\rangle$
 $\cong b_i$

example: Let $N = 3$.



9. Column vectors and row vectors (notation)

One way to represent vectors is in terms of their expansion coefficients in a given basis:

examples:

"bra" \nearrow $|B\rangle = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

"column" vector \nearrow $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
 $|A_1\rangle$ $|A_2\rangle$ $|A_3\rangle$

"ket" \curvearrowright $\langle B| = (b_1^*, b_2^*, b_3^*)$ \curvearrowleft "row" vector

the star here, again, is complex conjugation

Matrix multiplication (Part 1): How to multiply row vectors and column vectors

Rule: “rows into columns”: Multiply first element of row by first element of column, add product of second element of row and second element of column, add product of third element of row and third element of column, etc..

ex:

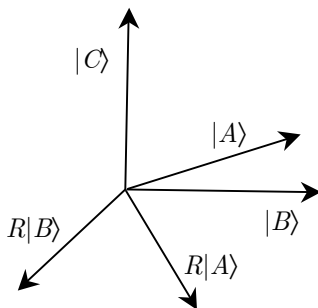
$$\langle A_1|B\rangle = (1, 0, 0) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (1 \times b_1) + (0 \times b_2) + (0 \times b_3) = b_1$$

II. Operators

1. A **linear operator** O is a map that assigns to any vector $|A\rangle$, another vector $O|A\rangle$, and that obeys the rule:

$$O(n|A\rangle + m|B\rangle) = n(O|A\rangle) + m(O|B\rangle) \quad \begin{array}{l} |B\rangle \text{ is any other vector and} \\ n, m \text{ are any numbers} \end{array}$$

example: Rotation operators. Let R be a linear operator that assigns to any vector another one that is rotated clockwise by 90° about vector $|C\rangle$.



2. Matrix representation of linear operators

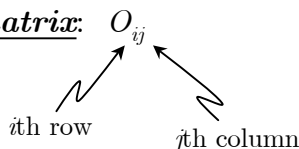
One way to represent an operator is in terms of its *components* in a given basis.

The **components** O_{ij} of an operator O in the basis $|A_1\rangle, |A_2\rangle, \dots, |A_N\rangle$ are defined by:

$$O_{ij} \equiv \langle A_i|O|A_j\rangle$$

These are *numbers*: the result of taking the inner-product of the *vectors* $|A_i\rangle$ and $O|A_j\rangle$.

The components of a linear operator form a **matrix**: O_{ij}



example: 2-dimensional operator O in $|A_1\rangle, |A_2\rangle$ basis.

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \quad \swarrow \quad 2 \times 2 \text{ matrix}$$

Column and row vectors are special cases of matrices: A column vector is a matrix with a single column; a row vector is a matrix with a single row.

Matrix multiplication (Part 2): How to multiply matrices

To multiply two matrices A and B :

- (1) **First check:** The number of columns of A must be equal to the number of rows of B .
- (2) Given an $n \times m$ matrix A and an $m \times r$ matrix B , their product is an $n \times r$ matrix C with entries given by:

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \dots + A_{im}B_{mj} \quad \leftarrow \text{general "row into column" rule}$$

example:

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \quad |B\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

2×2 matrix 2×1 matrix

can form their product $O|B\rangle$, which is a 2×1 matrix:

$$O|B\rangle = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} (O_{11}b_1 + O_{12}b_2) \\ (O_{21}b_1 + O_{22}b_2) \end{pmatrix}$$

3. Eigenvectors and Eigenvalues

An eigenvector of an operator O is a vector $|\lambda\rangle$ that does not change its direction when O acts on it. In other words, the result of acting with O on one of its eigenvectors is just to multiply the eigenvector by a number (which at most changes its length, but not its direction). This number is called an eigenvalue of O .

$$O|\lambda\rangle = \lambda|\lambda\rangle$$

λ is a number: an eigenvalue of O

$|\lambda\rangle$ and $\lambda|\lambda\rangle$ are two *different* vectors. They point in the same direction but have different lengths.

Note: By convention we usually label eigenvectors by their associated eigenvalues. So the eigenvector associated with λ is called $|\lambda\rangle$. But of course λ and $|\lambda\rangle$ are different mathematical objects: one is a number and one is a vector.

example: Let O be a 4-dim operator with matrix representation in a particular basis given by:

$$O = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

Then it has 4 eigenvectors given by:

$$|A\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |B\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |C\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |D\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Check:

$$O|A\rangle = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cong 5|A\rangle$$

Similarly: $O|B\rangle = 3/2|B\rangle$, $O|C\rangle = 2|C\rangle$, $O|D\rangle = -7|D\rangle$

We say: “ $|A\rangle$ is an eigenvector of O with eigenvalue 5, $|B\rangle$ is another eigenvector of O with eigenvalue 3/2”, etc...”

4. Hermitian Operators

A Hermitian operator is a linear operator that only has *real numbers* as eigenvalues.

(This is important, since we are going to associate eigenvalues with the values of measurable properties, and all measurements always yield real numbers.)

ASIDE: Some important mathematical characteristics of Hermitian operators:

- (1) Eigenvectors of a Hermitian operator (that don't share the same eigenvalue) are all mutually orthogonal.
 - (2) A Hermitian operator will always have at least one set of eigenvectors that form a basis for the vector space it is defined on.
 - (3) If a Hermitian operator on an N -dim vector space has N different eigenvalues, then for each there is a unique (up to length) eigenvector.
 - (4) Any vector in a given vector space is an eigenvector for some (*complete*) Hermitian operator defined on that space.
- A complete operator has eigenvalues that are all different.*