13a. Quantum Probabilities and Interference

- Classical probability theory is based on classical (Boolean) logic.
- The probabilities defined by the Born Rule in QM are based on quantum (non-Boolean) logic.
- **Consequence:** QM probabilities do not satisfy the classical Or-Addition Rule.

1. Classical Probability Theory and the Classical Or-Addition Rule

- A classical probability theory is given by a triple \((\Omega, \mathcal{F}, \Pr_c)\) where:
  - \(\Omega\) is a set of simple events (the sample space).
  - \(\mathcal{F}\) is a set of compound events obtained by taking all possible combinations of the simple events using set-theoretic complement and union.
  - \(\Pr_c\) is a probability function that maps elements of \(\mathcal{F}\) to \([0, 1]\) and satisfies the following axioms:

\[
\begin{align*}
(C1) \quad & \Pr_c(\emptyset) = 0 \\
(C2) \quad & \Pr_c(\neg A) = 1 - \Pr_c(A) \\
(C3) \quad & \Pr_c(A \cup A') = \Pr_c(A) + \Pr_c(A') - \Pr_c(A \cap A')
\end{align*}
\]
Example:
- Let \( \Omega = \{1, 2, 3, 4, 5, 6\} \) represent all possible results of a single roll of a die.
- Form \( \mathcal{F} \) by taking all possible combinations of these events using complement and union: \{1\}, \{2\}, ..., etc., \{1\} \cup \{2\}, \{1\} \cup \{3\}, ..., etc., \neg\{1\}, \neg\{2\}, etc...
- Define the probability function to be \( \Pr_c(\{i\}) = 1/6 \), for \( i = 1...6 \). \textit{(Principle of Indifference)}
- \textbf{Then}: The probability of getting either 1 or 3 on a single roll is given by:

\[
\Pr_c(\{1\} \cup \{3\}) = \Pr_c(\{1\}) + \Pr_c(\{3\}) - \Pr_c(\{1\} \cap \{3\}) \tag{C3}
\]
\[
= 1/6 + 1/6 - 0 = 1/3
\]
- \textbf{And}: The probability of getting either a value in the range \{1, 2, 3\} or a value in the range \{3, 4, 5\} on a single roll is:

\[
\Pr_c(\{1, 2, 3\} \cup \{3, 4, 5\}) \\
= \Pr_c(\{1, 2, 3\}) + \Pr_c(\{3, 4, 5\}) - \Pr_c(\{1, 2, 3\} \cap \{3, 4, 5\}) \tag{C3} \\
= [\Pr_c(\{1\}) + \Pr_c(\{2\}) + \Pr_c(\{3\})] + [\Pr_c(\{3\}) + \Pr_c(\{4\}) + \Pr_c(\{5\})] - \Pr_c(\{3\}) \\
= [1/6 + 1/6 + 1/6] + [1/6 + 1/6 + 1/6] - 1/6 = 5/6
2. Quantum Probability Theory and "Interference"

- Replace the classical sample space $\Omega$ with a Hilbert space $\mathcal{H}$.
- Form a collection $\mathcal{L}$ of subspaces of $\mathcal{H}$ (the compound events) by taking all possible combinations of the rays in $\mathcal{H}$ (the simple events) using orthocomplement and linear span.
- Define a probability function $\text{Pr}_q$ on $\mathcal{L}$ by the Born Rule.
- **So:** A quantum probability theory is a triple $(\mathcal{H}, \mathcal{L}, \text{Pr}_q)$, with $\text{Pr}_q$ defined by $\text{Pr}_q(|a\rangle, |\psi\rangle) = |\langle a|\psi\rangle|^2$, where $|a\rangle, |\psi\rangle \in \mathcal{H}$.

- **Main Result:** Quantum probabilities, so-defined, do not in general satisfy C3!
- They do satisfy the following (where $V, W$ are subspaces of $\mathcal{H}$ and $0$ is the "zero" subspace):

(Q1) $\text{Pr}_q(0) = 0$
(Q2) $\text{Pr}_q(V^\perp) = 1 - \text{Pr}_c(V)$
(Q3) $\text{Pr}_q(V \oplus W) = \text{Pr}_q(V) + \text{Pr}_q(W)$, when $V \perp W$

- **Recall:** Linear span $\oplus$ does not correspond to classical "or".
**Example:** 2-slit probabilities and interference

With Slit $A$ open,
\[ \Pr_q(e \text{ is at } x \text{ in state } \psi_A(x)) = |\psi_A(x)|^2 \]

With Slit $B$ open,
\[ \Pr_q(e \text{ is at } x \text{ in state } \psi_B(x)) = |\psi_B(x)|^2 \]
**Example:** 2-slit probabilities and interference

- With both slits open, the probability that $e$ is located at $x$ is $|\psi_A(x) + \psi_B(x)|^2$.
- This is *not* equal to $|\psi_A(x)|^2 + |\psi_B(x)|^2$, which, according to (C3), represents the probability that the electron *either* went through slit $A$ or slit $B$. 

Interference distribution (what happens)  \[ |\psi_A(x) + \psi_B(x)|^2 \]

A or B distribution (what doesn't happen)  \[ |\psi_A(x)|^2 + |\psi_B(x)|^2 \]
Let's see how this works using projection operators...

- **Recall:** The projection operator $P_{|a_i\rangle} = |a_i\rangle\langle a_i|$ corresponds to the 1-dim subspace defined by $|a_i\rangle$ (i.e., the ray in which $|a_i\rangle$ is pointing).

- **And:** $\sum_i P_{|a_i\rangle} = 1.$

**Def.** Suppose $Q$ is a linear operator on an $N$-dim vector space $\mathcal{H}$ with orthonormal basis $|b_1\rangle, \ldots, |b_N\rangle$. Then the trace $\text{Tr}(Q)$ of $Q$ is given by:

$$\text{Tr}(Q) \equiv \sum_{i=1}^N \langle b_i | Q | b_i \rangle = \langle b_1 | Q | b_1 \rangle + \langle b_2 | Q | b_2 \rangle + \cdots + \langle b_N | Q | b_N \rangle$$

- $\text{Tr}(Q) =$ the sum of the diagonal elements of any matrix representation of $Q$.
- All such representations have this sum in common: The trace is independent of the basis it's calculated in.

**Properties of the trace:**

- $\text{Tr}(\lambda A) = \lambda \text{Tr}(A)$ where $\lambda$ is any number
- $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$
- $\text{Tr}(AB) = \text{Tr}(BA)$
**The Born Rule in terms of projection operators:**

\[
\Pr_q(\text{value of } A \text{ is } a_i \text{ in state } |\psi\rangle) = |\langle a_i|\psi\rangle|^2
\]

\[
= \langle \psi|a_i\rangle\langle a_i|\psi\rangle
\]

\[
= \sum_j \langle \psi|P_{a_j}|a_i\rangle\langle a_i|\psi\rangle \quad \text{where } \sum_j P_{a_j} = 1
\]

\[
= \sum_j \langle \psi|a_j\rangle\langle a_j|a_i\rangle\langle a_i|\psi\rangle
\]

\[
= \sum_j \langle a_j|a_i\rangle\langle a_i|\psi\rangle\langle \psi|a_j\rangle
\]

\[
= \text{Tr}(|a_i\rangle\langle a_i|\psi\rangle\langle \psi|)
\]

\[
= \text{Tr}(P_{a_i}P_{|\psi\rangle})
\]

- **So:** \( \Pr_q(\text{value of } A \text{ is } a_i \text{ in state } |\psi\rangle) = \text{Tr}(P_{a_i}P_{|\psi\rangle}) \)

- \( P_{a_i} \) is the projection operator corresponding to the state \( |a_i\rangle \)
  (more precisely, the 1-dim subspace defined by \( |a_i\rangle \)).

- \( P_{|\psi\rangle} \) is the projection operator corresponding to the state \( |\psi\rangle \)
  (more precisely, the 1-dim subspace defined by \( |\psi\rangle \)).

- The projection operator corresponding to a state is called the **statistical operator** (or, **density matrix**) for the state.
• Consider composite state of *Hardness* measuring device and *black* electron:
\[ |\psi\rangle = \sqrt{\frac{1}{2}} \left\{ |"h"\rangle |h\rangle + |"s"\rangle |s\rangle \right\} \]

• It's statistical operator \( P_{|\psi\rangle} = |\psi\rangle \langle \psi | \) is given by:
\[
P_{|\psi\rangle} = \frac{1}{2} \left\{ |"h"\rangle \langle h | + |"s"\rangle \langle s | \right\} \left\{ |"h"\rangle \langle h | + |"s"\rangle \langle s | \right\}
\]
\[
= \frac{1}{2} \left\{ |"h"\rangle \langle h | + |"s"\rangle \langle s | \right\} \left\{ |"h"\rangle \langle h | + |"s"\rangle \langle s | \right\} + \frac{1}{2} \left\{ |"h"\rangle \langle s | \right\} \left\{ |s\rangle \langle h | \right\}
\]
\[
= \frac{1}{2} \left\{ P_{|s\rangle} \otimes P_{|h\rangle} + P_{|h\rangle} \otimes P_{|s\rangle} \right\} + \frac{1}{2} \langle |s\rangle \otimes |h\rangle \rangle \langle |h\rangle \otimes |s\rangle \rangle
\]

\[ \text{stat operator for } |"h"\rangle |h\rangle \text{ stat operator for } |"s"\rangle |s\rangle \text{ interference terms!} \]

• \( \Pr_q(\text{value of } A \text{ is } a_i \text{ in state } |\psi\rangle) = \text{Tr}(P_{|a_i\rangle} P_{|\psi\rangle}) \)
\[
= \text{Tr} \left( \frac{1}{2} P_{|a_i\rangle} P_{|s\rangle} \otimes P_{|h\rangle} \right) + \text{Tr} \left( \frac{1}{2} P_{|a_i\rangle} P_{|h\rangle} \otimes P_{|s\rangle} \right) + \text{Tr} \left( \frac{1}{2} P_{|a_i\rangle} |"h"\rangle \langle s | \otimes | h\rangle \langle s | \right) + \text{Tr} \left( \frac{1}{2} P_{|a_i\rangle} |"s"\rangle \langle h | \otimes | s\rangle \langle h | \right)
\]
\[
= \Pr_q(\text{value of } A \text{ is } a_i \text{ in state } |"h"\rangle |h\rangle) + \Pr_q(\text{value of } A \text{ is } a_i \text{ in state } |"s"\rangle |s\rangle)
\]
+ interference terms
13b. Decoherence and Interference

• **Claim**: When an observer ends up in an entangled state with a measuring device, environmental interactions destroy interference effects and *decohere* the state into one associated with a definite measurement outcome.

• Let $|\text{hard}\rangle_E, |\text{soft}\rangle_E$ be states of the environment $E$ in which it's correlated with a hard electron and a soft electron, respectively.

• **Then**: It is experimentally impossible to distinguish between:

\[
(1) \text{ The state } \sqrt{\frac{1}{2}} \left\{ |"\text{hard}"\rangle_m |\text{hard}\rangle_e |\text{hard}\rangle_E + |"\text{soft}"\rangle_m |\text{soft}\rangle_e |\text{soft}\rangle_E \right\}.
\]

\[
(2) \text{ Either of the states: } |"\text{hard}"\rangle_m |\text{hard}\rangle_e |\text{hard}\rangle_E \text{ or } |"\text{soft}"\rangle_m |\text{soft}\rangle_e |\text{soft}\rangle_E.
\]

• **Recall**: To distinguish between (1) and (2), we would need a very complex multi-particle property that (1) possesses and that neither state in (2) possesses.

• Given that $E$ realistically has a huge number of degrees of freedom, it is experimentally impossible to measure such a property.

• So (1) and (2) are *indistinguishable* for all practical purposes!
13b. Decoherence and Interference

• **Claim:** When an observer ends up in an entangled state with a measuring device, environmental interactions destroy interference effects and *decohere* the state into one associated with a definite measurement outcome.

• Let |hard⟩_E, |soft⟩_E be states of the environment E in which it's correlated with a hard electron and a soft electron, respectively.

• **Then:** It is experimentally impossible to distinguish between:

\[
\begin{align*}
(1) & \quad \text{The state } \sqrt{\frac{1}{2}} \left\{ |"hard"⟩_m | hard⟩_e | hard⟩_E + |"soft"⟩_m | soft⟩_e | soft⟩_E \right\} . \\
(2) & \quad \text{Either of the states: } |"hard"⟩_m | hard⟩_e | hard⟩_E \text{ or } |"soft"⟩_m | soft⟩_e | soft⟩_E .
\end{align*}
\]

**What this is supposed to mean:**

• Whenever the post-measurement state of a composite system is of the form of (1), it *does*, for all practical purposes, describe a situation in which a definite measurement outcome occurred.

• The environment, *for all practical purposes*, collapses the entangled superposition.
Let's see how this is supposed to work using statistical operators...

- The statistical operator \( P_{\psi} = |\psi\rangle\langle\psi| \) for the state in (1) is:

\[
P_{\psi} = \frac{1}{2} \left\{ |"h"\rangle \langle h| E_h + |"s"\rangle \langle s| E_s \right\} \left\{ |"h"\rangle \langle h| E_h \right\} + \frac{1}{2} \left\{ |"s"\rangle \langle s| E_s \right\} \left\{ |"s"\rangle \langle s| E_s \right\} + \text{(interference terms)}
\]

\[
= \frac{1}{2} |"h"\rangle \langle h| E_h \left\{ |"h"\rangle \langle h| E_h \right\} + \frac{1}{2} |"s"\rangle \langle s| E_s \left\{ |"s"\rangle \langle s| E_s \right\} + \text{(interference terms)}
\]

\[
= \frac{1}{2} \left\{ |"h"\rangle \langle h| E_h \right\} \left\{ |"h"\rangle \langle h| E_h \right\} + \frac{1}{2} \left\{ |"s"\rangle \langle s| E_s \right\} \left\{ |"s"\rangle \langle s| E_s \right\} + \text{(interference terms)}
\]

\[
= \frac{1}{2} P_{|"h"\rangle} \otimes P_{|h\rangle} \otimes P_{|E_h\rangle} + \frac{1}{2} P_{|"s"\rangle} \otimes P_{|s\rangle} \otimes P_{|E_s\rangle}
\]

\[
+ \frac{1}{2} |"h"\rangle \langle h| E_h \left\{ |"h"\rangle \langle h| E_h \right\} + \frac{1}{2} |"s"\rangle \langle s| E_s \left\{ |"s"\rangle \langle s| E_s \right\} + \text{(interference terms)}
\]

\[
= \left( \text{stat operator for } |"h"\rangle |h\rangle |E_h\rangle \right) + \left( \text{stat operator for } |"s"\rangle |s\rangle |E_s\rangle \right) + \text{(interference terms)}
\]

- **Now:** Take the "partial trace" of \( P_{\psi} \) with respect to the Environment basis:

\[
Tr_E \left( P_{\psi} \right) \equiv \langle E_h | P_{\psi} | E_h \rangle + \langle E_s | P_{\psi} | E_s \rangle
\]

\[
= \frac{1}{2} P_{|"h"\rangle} \otimes P_{|h\rangle} + \frac{1}{2} P_{|"s"\rangle} \otimes P_{|s\rangle}
\]

- **So:** "Tracing over the environment" kills the interference terms!
Does decoherence solve the measurement problem?

- **No!**
- When we "trace over the environment", we're left with the statistical operator

\[ \frac{1}{2} P_{|\text{"hard"}\rangle} \otimes P_{|\text{hard}\rangle} + \frac{1}{2} P_{|\text{"soft"}\rangle} \otimes P_{|\text{soft}\rangle} \]

- This is the statistical operator for a "mixed state": This is how the state of a system is represented when its exact form is known only to lie within a set of possible states.

- In this case, the state of the system is *either* of the pair

\[ \{|\text{"hard"}\rangle_m|\text{hard}\rangle_e, |\text{"soft"}\rangle_m|\text{soft}\rangle_e \} \]

each with equal weight 1/2.

- **But:** The result of a measurement (as given by the **Projection Postulate** and by our experience) is a *definite* outcome. In this case, the result is *either* $|\text{"hard"}\rangle_m|\text{hard}\rangle_e$ or $|\text{"soft"}\rangle_m|\text{soft}\rangle_e$. It's definitely one of these two alternatives. It's not a weighted sum of them both!
13c. Consistent Histories

**Def 1.** A history $h$ is a time-indexed sequence of facts, represented by time-indexed projection operators:

$$h = (P_1(t_1), P_2(t_2), ..., P_n(t_n)).$$

- $P_1(t_1)$ might be $P_{\text{hard}}(t_1)$ which represents the property, at time $t_1$, "The value of Hardness is hard".

- Or: It might be $P_{\{a\}}(t_1)$ which represents the property, at time $t_1$, "The value of the property $A$ is $a$".

- Projection operators evolve via the Schrödinger dynamics:

  $$P(t) = e^{iHt/\hbar}P(0)e^{-iHt/\hbar}.$$
Def 2. The probability associated with a history $h$ is given by:

$$\Pr_q(h) = \text{Tr}(P_n(t_n)...P_2(t_2)P_1(t_1)P_{\psi}P_1(t_1)P_2(t_2)...P_n(t_n))$$

where $P_{\psi}$ is the statistical operator associated with an initial state $|\psi\rangle$.

- All the terms inside a trace commute with each other, so:
  $$\Pr_q(h) = \text{Tr}(P_n(t_n)P_n(t_n)...P_2(t_2)P_2(t_2)P_1(t_1)P_1(t_1)P_{\psi}).$$

- Since projection operators are idempotent, this is equal to:
  $$\Pr_q(h) = \text{Tr}(P_n(t_n)...P_2(t_2)P_1(t_1)P_{\psi}).$$

- And: This can be thought of as the trace version of the Born Rule for the probability that the system in the state $|\psi\rangle$, has the "historical property" represented by the operator $P_n(t_n)...P_2(t_2)P_1(t_1)$. 
**Def 3.** A family of histories is a time-indexed sequence of sets of "exhaustive" facts:

\[(\{P_1^{\alpha_1}(t_1)\}, \{P_2^{\alpha_2}(t_2)\}, ..., \{P_n^{\alpha_n}(t_n)\})\]

where each index \(\alpha_i = 1, ..., N\) and \(\{P_i^{\alpha_i}(t_i)\} = \{P_i^1(t_i), P_i^2(t_i), ..., P_i^N(t_i)\}\), such that

\[P_i^1(t_i) + P_i^2(t_i) + ... + P_i^N(t_i) = I_N.\]

- The projection operators in any set \(\{P_i^{\alpha_i}(t_i)\}\) represent all the possible values of the property associated with \(P_i(t_i)\).


Histories can be embedded in families of histories:

\[ t = t_n \quad P_n^1(t_n) \quad P_n^2(t_n) \quad \ldots \quad \ldots \quad P_n^N(t_n) \quad \{ P_n^{\alpha_n}(t_n) \} \]

\[ t = t_2 \quad P_2^1(t_2) \quad P_2^2(t_2) \quad \ldots \quad \ldots \quad P_2^N(t_2) \quad \{ P_2^{\alpha_2}(t_2) \} \]

\[ t = t_1 \quad P_1^1(t_1) \quad P_1^2(t_1) \quad \ldots \quad \ldots \quad P_1^N(t_1) \quad \{ P_1^{\alpha_1}(t_1) \} \]

\[ t = 0 \quad P_{|\psi\rangle} \quad \{ P_{|\psi\rangle} \} \]

\[ h = (P_{|\psi\rangle}, P_1^1(t_1), P_2^2(t_2), \ldots, P_n^1(t_n)) \]

\[ h' = (P_{|\psi\rangle}, P_1^2(t_1), P_2^N(t_2), \ldots, P_n^2(t_n)) \]

\[ h \text{ and } h' \text{ are two histories within the family } (\{ P_{|\psi\rangle} \}, \{ P_1^{\alpha_1}(t_1) \}, \ldots, \{ P_n^{\alpha_n}(t_n) \}). \]
- Histories can be embedded in families of histories:

\[
\begin{align*}
t = t_n & \quad P_n^1(t_n) \quad P_n^2(t_n) \quad \ldots \quad \ldots \quad P_n^N(t_n) \quad \{P_n^{\alpha_n}(t_n)\} \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
t = t_2 & \quad P_2^1(t_2) \quad P_2^2(t_2) \quad \ldots \quad \ldots \quad P_2^N(t_2) \quad \{P_2^{\alpha_2}(t_2)\} \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
t = t_1 & \quad P_1^1(t_1) \quad P_1^2(t_1) \quad \ldots \quad \ldots \quad P_1^N(t_1) \quad \{P_1^{\alpha_1}(t_1)\} \\
t = 0 & \quad P_{|\psi\rangle} \quad \{P_{|\psi\rangle}\}
\end{align*}
\]

- We can assign probabilities to histories within a family by means of Def. 2.
- These are quantum probabilities that exhibit interference effects.
- Are there histories within a given family that can be assigned classical probabilities?
• Are there histories within a given family that do not interfere with each other?

• These would be histories \( h, h' \) whose probabilities obey the classical Or-Addition Rule:

\[
\Pr_q(h \text{ or } h') = \Pr_q(h) + \Pr_q(h').
\]

• \textit{First}: Need an expression for the disjunction, \( h \text{ or } h' \), of two histories \( h, h' \).

\textit{Simple case}: Suppose \( h_A \) and \( h_B \) are histories that differ \textit{only} in the property at \( t = t_i \):

\[
\begin{align*}
    h_A &= (P_1(t_1), \ldots, P_i^A(t_i), \ldots, P_n(t_n)) \\
    h_B &= (P_1(t_1), \ldots, P_i^B(t_i), \ldots, P_n(t_n))
\end{align*}
\]

• Now let the history, \( h_A \text{ or } h_B \), be given by:

\[
\begin{align*}
    h_A \text{ or } h_B &= (P_1(t_1), \ldots, P_i^A(t_i) + P_i^B(t_i), \ldots, P_n(t_n))
\end{align*}
\]
Simple case, continued:

- $h_A$ or $h_B = (P_1(t_1), ..., P_i(t_i) + P_i(t_i), ..., P_n(t_n))$

- So: For an initial state $|\psi\rangle$:

$$\text{Pr}_q(h_A \text{ or } h_B)$$

$$= \text{Tr}(P_n(t_n)...)P_i(t_i)...P_1(t_1)...[P_i(t_i) + P_i(t_i)]...P_n(t_n))$$

$$= \text{Tr}(P_n(t_n)...P_i(t_i)...P_1(t_1)...P_1(t_1)...P_i(t_i)...P_n(t_n))$$

$$+ \text{Tr}(P_n(t_n)...P_i(t_i)...P_1(t_1)...P_1(t_1)...P_i(t_i)...P_n(t_n))$$

$$+ \{\text{Tr}(P_n(t_n)...P_i(t_i)...P_1(t_1)...P_1(t_1)...P_i(t_i)...P_n(t_n))$$

$$+ \text{Tr}(P_n(t_n)...P_i(t_i)...P_1(t_1)...P_1(t_1)...P_i(t_i)...P_n(t_n))\}$$

$$= \text{Pr}_q(h_A) + \text{Pr}_q(h_B) + \{\text{interference terms}\}$$

- Which means: The probabilities assigned to $h_A$ and $h_B$ by Def. 2 will be classical ($i.e.$, will obey the classical Or-Addition Rule) just when the interference terms vanish.
• **Now**: Consider the general case of $h$, $h'$ differing on all properties:

$$h = (P_1(t_1), ..., P_n(t_n))$$

$$h' = (P_1'(t_1), ..., P_n'(t_n))$$

$$h \text{ or } h' = ([P_1(t_1) + P_1'(t_1)], ..., [P_i(t_i) + P_i'(t_i)], ..., [P_n(t_n) + P_n'(t_n)])$$

• The probabilities assigned to $h$ and $h'$ by *Def. 2* will be classical just when the general interference term vanishes:

$$\text{Tr}(P_n(t_n) ... P_1(t_1)P_{\psi}P_1'(t_1)...P_n'(t_n)) = 0$$

---

**Def. 4.** Two histories $h = (P_1(t_1), ..., P_n(t_n))$, $h' = (P_1'(t_1), ..., P_n'(t_n))$ are *consistent* just when $\text{Tr}(P_n(t_n) ... P_1(t_1)P_{\psi}P_1'(t_1)...P_n'(t_n)) = 0$.

---

**Def. 5.** A *consistent family of histories* is a family of histories such that any two histories embeddable in it are consistent.

• A *consistent family of histories* is a collection of histories that defines a classical sample space! You can assign classical probabilities to its members.
Def. 6.

(1) \( h \) is a fine-grained history just when all projection operators in \( h \) are 1-dim.

(2) \( h' \) is a coarse-graining of \( h \) just when some projection operators in \( h' \) are sums of projection operators in \( h \).

- Fine-grained histories cannot in general be assigned classical probabilities.
- Course-grained histories can be assigned *approximate* classical probabilities, and these get more classical as \( \text{Tr} (P_n(t_n) \ldots P_1(t_1) \langle \psi | P_1'(t_1) \ldots P_n'(t_n) ) \to 0 \).
- As \( \text{Tr} (P_n(t_n) \ldots P_1(t_1) \langle \psi | P_1'(t_1) \ldots P_n'(t_n) ) \to 0 \), such course-grained histories "decohere".
- *Coarse-graining* a family of histories corresponds to tracing out the environment.
- The environment interacts with the coarse-grained histories to damp out the interference effects, rendering the family approximately consistent.
**Def. 7.** Two histories $h = (P_1(t_1), ..., P_n(t_n)), h' = (P_1'(t_1), ..., P_n'(t_n))$ are *decoherent* just when $\text{Tr}(P_n(t_n)...P_1(t_1)P_1'(t_1)...P_n'(t_n)) = 0$.

**Def. 8.** A *decoherent family of histories* is a family of histories such that any two histories embeddable in it are decoherent.

**Characteristics of the Consistent/Decoherent Histories (CH) Approach**

- Replaces *states* of a physical system with *histories* a physical system.
- The properties (projection operators) that make up a history evolve *only via* the Schrödinger dynamics (no Projection Postulate).
- Identifies a way to associate a probability with a history (*Def. 2*).
- Identifies a condition that picks out those families of histories that are classical (or approximately classical) (*Defs. 4, 5*).
**Problems**

1. **How are alternative histories within a decoherent family to be interpreted?**
   - Is one history actual and the others just possible?
   - Or do all histories within a decoherent family occur? If so, then how are probabilities explained? (This is the *Problem of Probabilities* that Many Worlds faces.)

2. **How are alternative decoherent families to be interpreted?**
   - Any history $h$ can be embedded in many different mutually incompatible decoherent families (any one of which defines an approximately classical probability space).
   - Which do we choose in order to calculate the probability of $h$? (This is the *Preferred Basis Problem* that Many Worlds faces.)

**Problems 1 & 2 Combined:**
- Seem to indicate that CH isn't fundamentally different from Many Worlds.
- All CH does is replace world-talk with history-talk, and adds a criterion for identifying histories that behave "classically".
3. **General Problem with the Notion of decoherence**

- "Tracing over the environment" (or "coarse-graining" histories) does not pick out a unique measurement/interaction outcome.
- It does not effect a "collapse" of superposed states (or "interfering" histories).
- So it cannot be appealed to in order to reconcile superpositions (or "interfering" histories) with our experience of unique outcomes.