11. Bohm's Theory (Bohmian Mechanics)

- **Motivation**: Replace Hilbert state space of QM with one that is more classical and reproduces QM predictions.

### 4 Principles of Bohmian Mechanics

1. **States**: The state of a physical system is given by both a wave function $\psi$ and particle positions.

- **Recall**: In QM, a state is given entirely by a wave function $\psi$.
- In classical mechanics, a state is given by positions $x$ and momenta $p$:
  - 1 particle needs 6 numbers: $(x, y, z; p_x, p_y, p_z) = 1$ point in 6-dim phase space.
  - $N$ particles need $6N$ numbers: $(x_1, y_1, z_1; p_1^x, p_1^y, p_1^z; \ldots; x_N, y_N, z_N; p_N^x, p_N^y, p_N^z) = 1$ point in $6N$-dim phase space.

**Examples of configuration space (state space of positions):**

- $q = (x, y, z)$
- $q' = (x', y', z')$
- $Q = (x_1, y_1, z_1, \ldots, x_N, y_N, z_N)$
- $Q' = (x_1', y_1', z_1', \ldots, x_N', y_N', z_N')$

Now add specification of $\psi$ for each point in configuration space and get the state space of BM!
2. **Wave Function Dynamics**: The wave function associated with a state evolves according to the Schrödinger dynamics:

\[
\psi(Q_i, t_i) \xrightarrow{t_i \rightarrow t_f} \psi(Q_f, t_f)
\]

3. **Particle Dynamics**: Particle velocities are determined by Bohm's Equation:

\[
\vec{V}_i[\psi(Q)] = \frac{dq_i}{dt} = \frac{\hbar}{m_i} \text{Im} \left\{ \frac{\psi \ast \vec{r}_i \psi}{\psi \ast \psi} \right\}_Q \approx \frac{\text{probability current}}{\text{probability density}}
\]

The velocity \( \vec{V}_i \) of the \( i \)th particle, located at \( q_i = (x_i, y_i, z_i) \)...

... is a function of its mass \( m_i \) and the \( N \)-particle wave function \( \psi(Q) \), which depends on the positions \( Q \) of all the \( N \) particles.
4. **The Distribution (or Statistical) Postulate:** At some time \( t_0 \), particle positions are given by a probability defined by the wave function at \( t_0 \):

\[
\text{Pr(} \text{particle positions are } Q \text{ at time } t_0 \text{)} = |\psi(Q, t_0)|^2
\]

- **What this entails:** BM is *empirically indistinguishable* from QM: it reproduces all the QM probability predictions.

- **QM says:** (*Born Rule*) The probabilities for particle positions at *any* time \( t \) are given by \( |\psi(Q, t)|^2 \).

- **BM says exactly the same thing, because:**
  - The probability density \( \rho = |\psi|^2 \) is *conserved* by the Schrödinger equation (the equation of continuity holds for \( \rho \): \( \partial \rho / \partial t + \text{div} J = 0 \)).
  - So: If at time \( t_0 \), the probabilities are given by \( |\psi(Q, t_0)|^2 \) (the BM Distribution Postulate), then at any future (or past) time \( t \), the probabilities will be given by \( |\psi(Q, t)|^2 \) (the QM Born Rule).
• **What Principles 2, 3, and 4 are saying:** The point \( Q \) (representing the positions of all the \( N \) particles at any given time) moves about configuration space by being "guided" by the wave function \( \psi \! \)!

\[
Q \xrightarrow{t_i \to t_f} Q' \quad \text{particle dynamics via } \psi
\]

• **One interpretation:** The particles are swept along by the probability current defined by \( \psi \) (just like charges that are swept along in an electrical current).

• **Recall 2-slit experiment:** Are the electrons really particles that are being guided by some force that makes them impact the screen in an interference pattern? (Bohm's Theory = "Pilot Wave" theory.)

  - **But:** This analogy is not perfect: \( \psi \) is a function on configuration space (6\( N \)-dim for \( N \) particles), not physical space (3-dim Euclidean space).
  - **So:** \( \psi \) literally *isn't* a physical force (like an electric field).
  - **But:** Maybe it encodes properties of a physical force.
Characteristics of Bohmian Mechanics

(A) Positions of particles are always determinate. (Particles always have definite positions.)

(B) Positions evolve completely deterministically. (Any initial position state \( Q \) evolves to a \textit{unique} final position state \( Q' \).)

(C) BM reproduces the same probability predictions as QM. 
   \textit{But:} In BM, probabilities are \textit{epistemic}! Particles \textit{always} have definite positions, and BM probabilities just reflect our ignorance as to what they are.
Measuring the Hardness of a black electron:

- Inside Hardness box, black wave function "splits" into soft and hard wave functions.
- Depending on where electron is initially located, it will either be carried up with the hard wave function, or down with the soft wave function.
- An initial position in upper half of the black wave function entails it gets carried up.
- \[ |black\rangle|\psi_a(x)\rangle \longrightarrow \sqrt{\frac{1}{2}}|hard\rangle|\psi_b(x)\rangle + \sqrt{\frac{1}{2}}|soft\rangle|\psi_c(x)\rangle \]
• **Now**: Start with *black* electron. First measure *Hardness*, then *Color*.

• **QM**: \( \Pr(\text{black}) = \Pr(\text{white}) = 1/2 \).

• **BM**: Electron's initial location determines what its final *Color* value will be:

- If a *black* electron is initially located in the top half of the *black* wave function, it has a 50/50 chance of being either in the upper top half or the lower top half.

- **So**: It has a 50/50 chance of emerging as a *black* electron out of the *Color* box.
• Now: Start with black electron. First measure Hardness, then Color.
• QM: $\Pr(\text{black}) = \Pr(\text{white}) = 1/2$.
• BM: Electron's initial location determines what its final Color value will be:

- If a black electron is initially located in the bottom half of the black wave function, it has a 50/50 chance of being either in the upper bottom half or the lower bottom half.
- So: It has a 50/50 chance of emerging as a black electron out of the Color box.
- Thus: There is a 50/50 chance of the black electron being black after the Color measurement, if all we know is that it is initially located somewhere in the black wave function.
• **Now**: Send *black* electrons through a 2-path device, *without barrier*.
• **QM**: 100% will emerge *black*.
• **BM**: 100% will emerge *black*.

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| $|\text{black}\rangle|\psi_0(x)\rangle$ | $e_1$ carried by hard wave function. |
|----------------------------------------|-------------------------------------|
| $|\text{black}\rangle|\psi_s(x)\rangle$ | $e_2$ carried by soft wave function. |

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So *all* electrons, no matter what their initial positions, get carried up with *black* wave function.
• **Now**: Send *black* electrons through a 2-path device, *with barrier*.
• **QM**: Of those that get through, 50% will be *black*, 50% will be *white*.
• **BM**: Of those that get through, 50% will be *black*, 50% will be *white*.

Suppose $e_1$ is in upper top half and $e_2$ is in lower half of black wave function.

![Diagram of electron path through a 2-path device with barriers and wave functions.](image)

Only $e_1$ gets through due to its initial location in top half of black wave function. Its position is now in upper half of hard wave function.

So it gets carried up by black wave function.

If $e_1$ was initially in bottom top half of black wave function, it would enter Color box in bottom half of hard wave function, and exit as a white electron!
**Contextual Properties**

- A property is *intrinsic* just when, whether or not a physical system possesses it does not depend on how it is measured.
- A property is *contextual* just when, whether or not a physical system possesses it depends on how it is measured.

- In BM, position is an intrinsic property; all others are contextual.
- *Ex:* In BM, *Hardness* is a contextual property.

- Electron starts out in same initial location.
- *And:* Depending on how it is measured, it's *Hardness* value will be either *hard* or *soft.*
**Locality**

- **Consider:** 2 electrons in an entangled state ($e_1$ at point $a$, $e_2$ at point $f$):
  \[
  \sqrt{\frac{1}{2}} |\text{hard}\rangle_1 |\psi_a(x)\rangle_1 |\text{soft}\rangle_2 |\psi_f(x)\rangle_2 - \sqrt{\frac{1}{2}} |\text{soft}\rangle_1 |\psi_a(x)\rangle_1 |\text{hard}\rangle_2 |\psi_f(x)\rangle_2
  \]

- Now measure **Hardness** of $e_1$.
- State becomes: $\sqrt{\frac{1}{2}} |\text{hard}\rangle_1 |\psi_b(x)\rangle_1 |\text{soft}\rangle_2 |\psi_f(x)\rangle_2 - \sqrt{\frac{1}{2}} |\text{soft}\rangle_1 |\psi_f(x)\rangle_1 |\text{hard}\rangle_2 |\psi_f(x)\rangle_2$
- Now measure **Hardness** of $e_2$.
- $e_2$ is carried down through **soft** exit (only **soft** wave function acts on it).
- If $e_1$ had not been measured, then $e_2$ would have come out **hard**, due to it's initial location!

\[
|\text{hard}\rangle_1 |\psi_b(x)\rangle_1 \quad |\psi_a(x)\rangle_1
\]
\[
|\text{soft}\rangle_1 |\psi_c(x)\rangle_1 \quad |\psi_f(x)\rangle_2
\]
• In Bohm's Theory, electrons *always* have a definite position, and the final position of $e_2$ is determined by the final position of $e_1$.

• **Suppose:** Alice and $e_1$ are very far from Bob and $e_2$.

• **Suppose:** Bob knows the initial positions of $e_1$ and $e_2$, and he gets the strange result that $e_2$ came out *soft* (when it should have come out *hard*, given it's initial location).

• **Then:** Bob knows that Alice way over there must have used a *hard*-side up Hardness box to measure $e_1$!

• **This allows Bob and Alice to send instantaneous signals to each other!**
**How to send an instantaneous message in BM:**

- **Suppose:** Alice desires to send Bob a message instructing him to push either Button A or Button B at some future time $t$.
- They share initial positions of their $e_1$ and $e_2$ and agree to the following:
  - If Alice wants Bob to push Button A, then before $t$ she orients her *Hardness* box so that a *Hardness* measurement will yield the value *hard*.
  - If Alice wants Bob to push Button B, then before $t$ she orients her *Hardness* box so that a *Hardness* measurement will yield the value *soft*.
- At $t$, Bob measures his electron: This will tell him what the outcome of Alice's measurement was, and hence which Button she wants him to push!

\[
|\text{hard}\rangle_1 |\psi_b(x)\rangle_1 \quad \text{h} \quad \text{H} \quad |\psi_a(x)\rangle_1 \quad \text{s} \\
|\text{soft}\rangle_1 |\psi_c(x)\rangle_1
\]
QM vs BM on instantaneous messaging:

- **Under a literal interpretation of QM:**
  - The outcome of an $e_2$ measurement depends non-locally on the outcome of an $e_1$ measurement.
  - **But:** The outcome of an $e_2$ measurement does *not* depend on whether or not an $e_1$ measurement was done.

- **In BM:**
  - The outcome of an $e_2$ measurement *does* depend on whether or not an $e_1$ measurement was done.

\[
\begin{align*}
\left| \text{hard} \right\rangle_1 & \left| \psi_b(x) \right\rangle_1 & \left| \text{soft} \right\rangle_1 & \left| \psi_c(x) \right\rangle_1 \\
& \left| \psi_a(x) \right\rangle_1 & & \\
& & H & \\
& & & \left| \psi_f(x) \right\rangle_2 \\
& & s & \\
& h & & \\
& \left| \psi_s(x) \right\rangle & & \\
& & \left| \psi_f(x) \right\rangle & \left| \psi_s(x) \right\rangle \\
& & h & \\
& & & \left| \text{soft} \right\rangle_1 & \left| \psi_c(x) \right\rangle_1 \\
\end{align*}
\]
**Claim:** For any given measurement set-up, the initial positions of particles can *never* be known in BM. *All* that can be known is the wave function.

**Does BM violate Special Relativity?**

- In Special Relativity, the simultaneity of distant events in the same inertial reference frame is relative: there is *no* absolute fact of the matter which occurs before the other.

- In BM, there *is* a fact of the matter (a "privileged" reference frame that determines the simultaneity of distant events).

  *Why?* Because, *if* they can instantaneous message, Alice and Bob will always agree on the order of their measurements.

- **So:** BM will violate special relativity, *unless it can explain why the privileged reference frame is in principle unobservable.*

  **Claim:** For any given measurement set-up, the initial positions of particles can *never* be known in BM. *All* that can be known is the wave function.

- **Thus:** *In practice*, instantaneous signaling is *not* possible in BM.

- **So:** *In practice*, the privileged simultaneity frame cannot be determined.

- **And so:** *In practice*, BM does not violate Special Relativity.
Why initial particle positions can never be known in BM:

- Consider measuring the Hardness of a black electron $e$:

- If we could determine $e$'s initial position, then we could predict with certainty which exit it will take:
  - Initially in upper half, then hard exit.
  - Initially in lower half, then soft exit.

- So: How could we determine initial position?

- Problem: According to BM, any attempt will change the pre-Hardness measurement wave function, and so affect all subsequent measurements!
• **Suppose:** Before measuring *Hardness* of $e$, we measure its position:

$$|\text{ready}_m\rangle|\psi_a(x)\rangle_e|\text{black}_e\rangle \longrightarrow \frac{\sqrt{1}}{2}|+\rangle_m|\psi_a^+(x)\rangle_e|\text{soft}_e\rangle + \frac{\sqrt{1}}{2}|\psi_a^-(x)\rangle_e|\text{black}_e\rangle$$

• If $e$ is measured to be in the *upper-half* of $\psi_a(x)$, then it's (effective) wave function is now $\psi_a^+(x)$.

• This will not allow us to predict how it will move through a *Hardness* device:

![Diagram](attachment:diagram.png)

• If $e$ is initially in upper-half of $\psi_a(x)$, then it will emerge from $m$ as $\psi_a^+(x)$.

• **But:** To predict where it will emerge from $H$, we need to know if it's in the *upper-half* or *lower-half* of $\psi_a^+(x)$!

• And to measure this is to disrupt the wave function again!