09a. Collapse

- **Recall:** There are two ways a quantum state can change:

1. In absence of measurement, states change *via Schrödinger dynamics*:

   \[
   |\psi(t_1)\rangle \xrightarrow{\text{Schrödinger evolution}} |\psi(t_2)\rangle
   \]

2. In presence of measurement, states change *via the Projection Postulate*:

   When a measurement of property \(B\) is made on a state \(|\psi\rangle = a_1|b_1\rangle + ... + a_N|b_N\rangle\) expanded in the eigenvector basis of \(B\) with result \(b_i\), then \(|\psi\rangle\) collapses to \(|b_i\rangle\):

   \[
   |\psi\rangle \xrightarrow{\text{collapse}} |b_i\rangle
   \]

- **Problems:**
  - What is a *measurement*?
  - *When* is the Projection Postulate supposed to take over from the Schrödinger dynamics?
Possible Responses:

1. When a conscious observer looks at a measuring device.
   • Consequence: Dualism -- 2 fundamentally different types of physical system.
     ○ Purely physical systems: Always evolve via Schrödinger dynamics.
     ○ Conscious systems: Interact with physical systems in certain situations to cause collapse via Projection Postulate.

2. When a macroscopic system interacts with a microscopic system.
   • Consequence: Dualism again -- 2 fundamentally different types of physical system.
     ○ Microscopic systems: Always evolve via Schrödinger dynamics.
     ○ Macroscopic systems: Interact with microscopic systems in certain situations to cause collapse via Projection Postulate.
Can experiments determine when a collapse occurs?

In principle, Yes!
But in practice, No!

Theory 1: Collapse occurs at $t_1$.
Prediction 1: At $t_1$, state will be:
either $|\text{"hard"}\rangle_m|\text{hard}\rangle_e$ or $|\text{"soft"}\rangle_m|\text{soft}\rangle_e$,
each with prob = 1/2.

Theory 2: Collapse occurs after $t_1$.
Prediction 2: At $t_1$, state will be:
$\sqrt{\frac{1}{2}} \left( |\text{"hard"}\rangle_m|\text{hard}\rangle_e + |\text{"soft"}\rangle_m|\text{soft}\rangle_e \right)$.

- Suppose the $m$-$e$ entangled state of Theory 2 has a measurable "2-particle"
  property that neither of the $m$-$e$ separable states of Theory 1 have.
- This would let us distinguish between Theory 1 and Theory 2.
• **Task:** Find a "2-particle" property of the entangled $m$-$e$ state of Prediction 1 that is not possessed by the separable $m$-$e$ states of Prediction 2.

• We have **Hardness** $H^e$ and **Color** $C^e$ operators that act on states of $e$:

\[
H^e|\text{hard}\rangle_e = +1|\text{hard}\rangle_e \quad C^e|\text{black}\rangle_e = +1|\text{black}\rangle_e
\]
\[
H^e|\text{soft}\rangle_e = -1|\text{soft}\rangle_e \quad C^e|\text{white}\rangle_e = -1|\text{white}\rangle_e
\]

• Now define properties and states for $m$.

  ○ Let "**Hardness**" operator $H^m$ represent the property of $m$ of pointing to either "ready", "hard", or "soft".

  ○ Let $|\text{ready}\rangle_m$, $|\text{hard}\rangle_m$, $|\text{soft}\rangle_m$ be states of $m$ that are eigenvectors of $H^m$.

  ○ Let "**Color**" operator $C^m$ represent property of $m$ of pointing to either "ready", "black", or "soft".

  ○ Let $|\text{ready}\rangle_m$, $|\text{black}\rangle_m$, $|\text{white}\rangle_m$ be states of $m$ that are eigenvectors of $C^m$. 
• Represent the states of \( m \) by column vectors:

\[
\begin{align*}
|\text{ready}\rangle_m &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
|\text{"hard"}\rangle_m &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
|\text{"soft"}\rangle_m &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
|\text{"black"}\rangle_m &= \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} \\
|\text{"white"}\rangle_m &= \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \end{pmatrix}
\end{align*}
\]

• \textbf{Then:}

\[
\begin{align*}
|\text{"hard"}\rangle_m &= \sqrt{\frac{1}{2}} (|\text{"black"}\rangle_m + |\text{"white"}\rangle) \\
|\text{"black"}\rangle_m &= \sqrt{\frac{1}{2}} (|\text{"hard"}\rangle_m + |\text{"soft"}\rangle) \\
|\text{"soft"}\rangle_m &= \sqrt{\frac{1}{2}} (|\text{"black"}\rangle_m - |\text{"white"}\rangle) \\
|\text{"white"}\rangle_m &= \sqrt{\frac{1}{2}} (|\text{"hard"}\rangle_m - |\text{"soft"}\rangle)
\end{align*}
\]

• Represent the \textit{"Hardness"} and \textit{"Color"} operators by:

\[
H^m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
C^m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

• \textbf{Then:}

\[
\begin{align*}
H^m|\text{ready}\rangle_m &= 0|\text{ready}\rangle_m \\
H^m|\text{"hard"}\rangle_m &= +1|\text{"hard"}\rangle_m \\
H^m|\text{"soft"}\rangle_m &= -1|\text{"soft"}\rangle_m \\
C^m|\text{ready}\rangle_m &= 0|\text{ready}\rangle_m \\
C^m|\text{"black"}\rangle_m &= +1|\text{"black"}\rangle_m \\
C^m|\text{"white"}\rangle_m &= -1|\text{"white"}\rangle_m
\end{align*}
\]
• The 2-particle operator $C^m \otimes I^e$ represents $m$'s "Color" in the 2-particle system.

• The 2-particle operator $I^m \otimes C^e$ represents $e$'s Color in the 2-particle system.

• \underline{And:} The 2-particle operator $(C^m \otimes I^e) - (I^m \otimes C^e)$ represents another 2-particle property, call it ("Color" – Color).

• \underline{Recall:} The final states of predicted by Theories 1 and 2 are given by

\[
|T1\rangle = \begin{cases} 
|"hard"\rangle_m |hard\rangle_e & \text{prob = 1/2} \\
|"soft"\rangle_m |soft\rangle_e & \text{prob = 1/2}
\end{cases}
\]

\[
|T2\rangle = \sqrt{\frac{1}{2}} \left( |"hard"\rangle_m |hard\rangle_e + |"soft"\rangle_m |soft\rangle_e \right).
\]

• \underline{Claim:} $|T2\rangle$ is an eigenstate of $(C^m \otimes I^e) - (I^m \otimes C^e)$ with eigenvalue 0, but $|T1\rangle$ in either form is \textit{not}!

• \underline{So:} At time $t_1$ we can (in principle) measure the property ("Color" – Color).
  \○ If Theory 2 is correct, the value should \textit{always} be 0.
  \○ If Theory 1 is correct, measurements should yield values other than 0.
• **But:** What happens if there is a single air molecule in the measuring device?

- Let $|\text{center}\rangle_a$ represent state of air molecule located underneath "hard" pointer-reading.
- Let $|\text{right}\rangle_a$ represent state of air molecule located underneath "soft" pointer-reading.

**Theory 1:** Collapse occurs at $t_1$.

**Prediction 1:** At $t_1$, state will be:

\[
\text{either } |\text{"hard"}\rangle_m |\text{hard}\rangle_e |\text{center}\rangle_a \text{ or } |\text{"soft"}\rangle_m |\text{soft}\rangle_e |\text{right}\rangle_a, \text{ each with prob } = 1/2.
\]

**Theory 2:** Collapse occurs after $t_1$.

**Prediction 2:** At $t_1$, state will be:

\[
\sqrt{\frac{1}{2}} |\text{"hard"}\rangle_m |\text{hard}\rangle_e |\text{center}\rangle_a + \\
\sqrt{\frac{1}{2}} |\text{"soft"}\rangle_m |\text{soft}\rangle_e |\text{right}\rangle_a.
\]

• These predictions are not eigenstates of ("Color" - Color)!

• To tell them apart, need a more complicated, 3-particle property of the $m$-$e$-$a$ system.

• If there are other microscopic systems in $m$, need even more complicated multi-particle properties.
09b. GRW Collapse Theory  
Ghirardi, Rimini, Weber (1986)

- **Motivation:** Most (all?) properties can be correlated with the *position* of a pointer in an appropriate measuring device.

- We can maintain Option A1 if we *stipulate* that (for whatever reason) superpositions of *position eigenstates* collapse with high probability.

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**GRW Dynamical Law**

During any time interval, there is a non-zero probability that the state of an elementary particle will collapse to a position eigenstate.

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- Suppose $|\psi\rangle$ represents the state of an elementary particle.
- Expand $|\psi\rangle$ in a basis of eigenvectors of position: $|\psi\rangle = a_1|x_1\rangle + \ldots + a_N|x_N\rangle$.
- The GRW Collapse Theory says:
  - $|\psi\rangle$ generally evolves *via* the Schrödinger dynamics.
  - But there is a non-zero chance that $|\psi\rangle$ will collapse to a position eigenstate.
  - If this occurs, the probability of it ending up in the specific position eigenstate $|x_i\rangle$ is given by the Born Rule:
    \[
    \Pr(\text{particle is located at position } x_i \text{ in state } |\psi\rangle) = |\langle \psi | x_i \rangle|^2 = |\psi(x_i)|^2
    \]
Why this helps:

Consider the entangled state of pointer particles and electron:

\[ \frac{\sqrt{1}}{2} \left( |x_1\rangle_1 |x_1\rangle_2 |x_1\rangle_3\ldots \right) |\text{hard}\rangle_e + \frac{\sqrt{1}}{2} \left( |x_2\rangle_1 |x_2\rangle_2 |x_2\rangle_3\ldots \right) |\text{soft}\rangle_e \]

Position states of pointer particles when pointer is located at the position, call it \( x_1 \), that registers "hard".

Position states of pointer particles when pointer is located at the position, call it \( x_2 \), that registers "soft".

- If just one of all of these (trillions of) pointer particles has a definite location, then according to the Projection Postulate, the superposition will collapse to the term containing that position eigenstate.

- So: Even if the probability of collapse in the GRW Law is extremely small, since any measuring device has trillions of particles, any superposition containing them will have an extremely likely chance of collapsing.

- Upshot: By modifying the Schrödinger dynamics with the GRW Law we guarantee that measuring devices will always have well-defined positions.
The GRW Law in terms of Wavefunctions:

- Let \( |\psi\rangle = a_1|x_1\rangle + ... + a_N|x_N\rangle \) represent the state of an elementary particle expanded in a basis of eigenvectors of position.
- \textbf{Recall:} The corresponding position wavefunction is defined by \( \psi(x) = \langle \psi | x \rangle \).
- \( \psi(x) \) encodes all the values of the expansion coefficients. \textit{Ex:} \( \psi(x_i) = a_i \).

- \textbf{Now:} Represent state vector collapse: \( |\psi\rangle \xrightarrow{\text{collapse}} |x_i\rangle \)

as wavefunction collapse: \( \psi(x) \xrightarrow{\text{collapse}} \psi(x_i) \)

\[ \text{general form of wavefunction} \]
\[ \text{for all position eigenstates} \]
\[ \text{specific form of wavefunction for state } |x_i\rangle \]

- Mathematically, wavefunction collapse is accomplished by multiplying the general wavefunction \( \psi(x) \) by a \textit{Dirac delta function} \( \delta(x - x_i) \):

\[ \psi(x_i) = \delta(x - x_i) \psi(x) \]
The Dirac Delta Function

- Think of $\delta(x - x_i)$ as a position eigenfunction.
- It is an infinite spike exactly at $x_i$, and zero everywhere else:

\[ \int \delta(x - x') dx' = 1. \]

\[ \int \delta(x - x') f(x) dx' = f(x'), \text{ for any } f(x). \]

So, more precisely, $\psi(x_i) = \int \delta(x - x_i) \psi(x) dx$.

Initial Problem:

- If our state is represented by an eigenvector of position, then according to Option A1 this means that it has a definite value of position (namely, $x_i$).
- So: Properties incompatible with position will be maximally indeterminate.

\[
\begin{array}{c}
\text{Uncertainty in position} = 0 \\
\Rightarrow \\
\text{Uncertainty in momentum} = \infty!
\end{array}
\]

- When a GRW collapse occurs (and we get an exact value of position), we could potentially have violations of conservation of momentum and energy!
**Technical Solution**

- For GRW collapses, instead of multiplying the wavefunction by a Dirac delta function $\delta(x - x_i)$, use a *Gaussian* (i.e., Bell-shaped) function $g_L(x - x_i)$ spread out a finite width $L$ about $x_i$.

  \[
g_L(x - x') = \frac{1}{\sqrt{\pi L^2}} e^{-\frac{(x - x')^2}{L^2}}
\]

  The Gaussian function *is* a legitimate function.
  - Centered at $x'$.
  - Width = $L$.
  - Maximum height = $\frac{1}{\sqrt{\pi L^2}}$.
  - Unit area.
  - $\lim_{L \to 0} g_L(x - x') = \delta(x - x_i)$.

- Can choose $L$ so that the uncertainty in momentum is effectively cut-off.

- **The price**: We've had to smear the position about $x_i$. 
Essential Characteristics of GRW Collapse Theory:

1. Modifies Schrödinger dynamics with GRW Law.
2. Keeps Projection Postulate.
3. Introduces two new constants of nature:
   (a) Probability per time per particle of collapse.
   (b) Width $L$ of Gaussian position eigenfunctions.
**Problems**

1. **Wavefunction Tails**
   - We just saw that GRW needs Gaussian position functions \( g_L(x - x_i) \) in order to avoid violations of energy/momentum conservation.
   - **But:** \( g_L(x - x_i) \) is never zero: It has non-vanishing tails.

![Graph](image)

- The tails asymptotically approach the \( x \)-axis, but never reach it, no matter how far away from \( x_i \) you get.
- \( g_L(x - x_i) \neq 0 \) for all finite values of \( x \).

- **So:** Even after a GRW collapse, an elementary particle is still in a superposition of position eigenvectors (there is still a non-zero probability of finding it at some location \( x \neq x_i \)).
- **Thus:** According to Option (1A), it still has no definite value of position! (When GRW agreed to use Gaussians instead of Dirac delta functions, they gave up exact position collapses.)
2. Positionless Measurements

- GRW assume that the position property is *fundamental*, in-so-far as all other properties must be correlated with the positions of pointers in measuring devices in order to measure them.

- *Is this correct?*

- *Ex:* Measuring a particle's *Hardness* by means of a fluorescent screen.
  - To measure *Hardness* of *P*, insert it into *Hardness* box.
  - If it's *hard*, it will exit at *h* and impact screen at *A*.
  - If it's *soft*, it will exit at *s* and impact screen at *B*.
• **Claim:** At no point in this process is the *Hardness* of the particle correlated with the *position* of a pointer (or any sort of measuring device).

• The impact of *P* at *A* or *B* correlates its *Hardness* with the *energies* (and not positions) of the electrons in the atoms of the fluorescent screen.

• **Suppose:** *P* is initially *black*.

• **Then:** Just after impact, its state can be represented by:

\[
\sqrt{\frac{1}{2}}|\text{hard}, x = A\rangle_p |ex\rangle_{e_1} ... |ex\rangle_{e_N} |unex\rangle_{e_{N+1}} ... |unex\rangle_{e_{2N}} \\
+ \sqrt{\frac{1}{2}}|\text{soft}, x = B\rangle_p |unex\rangle_{e_1} ... |unex\rangle_{e_N} |ex\rangle_{e_{N+1}} ... |ex\rangle_{e_{2N}}
\]

- A GRW collapse of any of the energy states of the electrons onto a position eigenstate will *not* cause the state of *P* to collapse into one or the other of these terms.
• What about the photons that the electrons emit? Don't they experience GRW collapses?

• No!

  ○ The GRW Collapse Law doesn't apply to relativistic objects (like photons).
  ○ Moreover: There can be just a few photons released on impact to record the Hardness measurement (the human eye can discern photons at very low intensities).
  ○ The GRW Collapse Law only guarantees that one of a large number (trillions!) of elementary particles will collapse for real times.

• So: To address this problem, GRW advocates will have to push back the process of measurement, perhaps to brain states in the human observer, to a point at which they can say the positions of something (brain neuron states?) get correlated with the Hardness property of $P$. 
3. Microscopic Measurements

- Due to the *randomness* of the GRW collapse, collapses will only occur in real times for *macroscopic* measuring devices (that have trillions of elementary particles).

- *But*: What about the possibility of *microscopic* measuring devices?

- These would *not* be expected to have GRW collapses in real times.

- If they could be correlated with macroscopic measuring devices, we would have the measurement problem all over again!