03. Vectors and Operators

I. Vectors and Vector Spaces: 9 Easy Steps

1. Vectors

- A vector = a magnitude ("length") and a direction
- One way to represent vectors:

![Diagram showing a vector in the x-y plane with components and an associated point (x, y).]

- Every point in x-y plane has a vector $|A\rangle$ associated with it.
- In this example, the collection of all vectors $|A\rangle$, $|B\rangle$, $|C\rangle$, ... forms a 2-dimensional vector space, call it $V$. 
2. **Vector addition**

- To add vectors $|A\rangle$ and $|B\rangle$ in $V$, place tail of $|B\rangle$ to head of $|A\rangle$ to form a third vector in $V$, $|A\rangle + |B\rangle$, whose tail is the tail of $|A\rangle$ and whose head is the head of $|B\rangle$:

![Vector addition diagram](image)

3. **Scalar (number) multiplication**

- *Numbers* can be multiplied to vectors. The result is another vector.

**Ex:** $5|A\rangle$ is a vector in $V$ that you get by multiplying the vector $|A\rangle$ by the number 5.
4. **Inner (or dot) product**

- Vectors can be multiplied to each other.
- One type of vector multiplication is called the *inner-product*.
- The inner-product of two vectors $|A\rangle$, $|B\rangle$ is written as $\langle A|B\rangle$ and is a number defined by:

$$\langle A|B\rangle = |A||B| \cos \theta$$

*Note:* This means $\langle A|A\rangle = |A||A| \cos(0) = |A|^2$.

So the *length* of a vector $|A\rangle$ is given by $|A| = \sqrt{\langle A|A\rangle}$. 
5. **Orthogonality**
- Two non-zero-length vectors $|A\rangle$ and $|B\rangle$ are *orthogonal* (perpendicular) just when their inner-product is zero: $\langle A|B \rangle = 0$.

**Check:** $\langle A|B \rangle = |A||B|\cos\theta = 0$, just when $\theta = 90^\circ$, given $|A| \neq 0 \neq |B|$.

6. **Dimensionality**
- The *dimension* of a linear vector space is equal to the maximum number $N$ of mutually orthogonal vectors.
7. **Orthonormal bases**

- An *orthonormal basis* of an $N$-dimensional vector space is a set of $N$ mutually orthogonal vectors, each with unit length (or *norm*).

**Note:** An $N$-dim vector space can have many different orthonormal bases!

**Ex:** Let $N = 3$.

Then $|A_1\rangle$, $|A_2\rangle$, $|A_3\rangle$ and $|A_1\rangle$, $|A_2\rangle$, $|A_3\rangle$ are two different orthonormal bases:
8. *Expansion in an orthonormal basis*

- In an $N$-dimensional vector space, any vector $|B\rangle$ can be *expanded* in terms of any orthonormal basis.

- **Suppose:** $|A_1\rangle, |A_2\rangle, \ldots, |A_N\rangle$ are $N$ orthonormal basis vectors.
- **Suppose:** $|B\rangle$ is an $N$-dim vector.
- **Then:** $|B\rangle = b_1|A_1\rangle + b_2|A_2\rangle + \ldots + b_N|A_N\rangle$.
- **And:** The numbers $b_i = \langle A_i | B \rangle$, $i = 1, \ldots, N$, are called *expansion coefficients*.

**Ex:** Let $N = 3$. 

\[ |B\rangle = b_1|A_1\rangle + b_2|A_2\rangle + b_3|A_3\rangle \]
9. **Column vectors and row vectors**

- One way to represent vectors is in terms of columns or rows of their expansion coefficients in a given basis:

\[
|B\rangle = b_1|A_1\rangle + b_2|A_2\rangle + b_3|A_3\rangle
\]

**Ex:**

\[
|B\rangle = \begin{pmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{pmatrix}
= b_1 \begin{pmatrix}
    1 \\
    0 \\
    0
\end{pmatrix} + b_2 \begin{pmatrix}
    0 \\
    1 \\
    0
\end{pmatrix} + b_3 \begin{pmatrix}
    0 \\
    0 \\
    1
\end{pmatrix}
\]

**Rule:** To turn a column vector into a row vector, take the complex conjugate of its expansion coefficients:

\[
\langle B | = b_1^*\langle A_1 | + b_2^*\langle A_2 | + b_3^*\langle A_3 |
\]

\[
\langle B | = (b_1^*, b_2^*, b_3^*) = b_1^*(1, 0, 0) + b_2^*(0, 1, 0) + b_3^*(0, 0, 1)
\]
II. Operators: 3 Easy Steps

1. Linear Operators

- A *linear operator* $O$ is a map that assigns to any vector $|A\rangle$, another vector $O|A\rangle$, such that, for any other vector $|B\rangle$ and numbers $n, m$,

$$O(n|A\rangle + m|B\rangle) = n(O|A\rangle) + m(O|B\rangle)$$

*Ex:* Rotation operators.
A map $R$ that assigns to any vector another one that is rotated clockwise by 90° about vector $|C\rangle$, is a linear operator.
2. **Matrix representation of linear operators**

- An operator can be represented by its *components* in a given basis.
- The *components* $O_{ij}$ ($i, j = 1, \ldots, N$) of an operator $O$ in the basis $|A_1\rangle, \ldots, |A_N\rangle$ are defined by:

  $$O_{ij} = \langle A_i | O | A_j \rangle$$

  **Note:** These are *numbers*: the result of taking the inner-product of the vectors $|A_i\rangle$ and $O|A_j\rangle$.

- The components $O_{ij}$ of a linear operator form the elements of a *matrix*.

  $i$th row $j$th column

  **Ex:** 2-dim operator $O$ in $|A_1\rangle, |A_2\rangle$ basis.

  $$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

- Column and row vectors are special cases of matrices: A column vector is a matrix with a single column; a row vector is a matrix with a single row.
Matrix multiplication

- To multiply two matrices $A$ and $B$:

  1. The number of *columns* of $A$ must be equal to the number of *rows* of $B$.

  2. *Row into Column Rule*: If $A$ is an $n \times m$ matrix and $B$ is an $m \times r$ matrix, their product is an $n \times r$ matrix $C$ with entries given by:

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{im}B_{mj}$$

**Ex.**

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \quad |B\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\quad 2 \times 2 \quad 2 \times 1$

$$O|B\rangle = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} O_{11}b_1 + O_{12}b_2 \\ O_{21}b_1 + O_{22}b_2 \end{pmatrix}$$

$\quad 2 \times 1$
3. **Eigenvectors and Eigenvalues**

- An *eigenvector* of an operator $O$ is a vector $|\lambda\rangle$ that does not change its direction when $O$ acts on it: $O|\lambda\rangle = \lambda|\lambda\rangle$, for some number $\lambda$.
- An *eigenvalue* $\lambda$ of an operator $O$ is the number that results when $O$ acts on one of its eigenvectors.
- **Convention:** Eigenvectors $|\lambda\rangle$ are labeled by their eigenvalues $\lambda$.

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**Note:** $\lambda$ and $|\lambda\rangle$ are different mathematical objects: $\lambda$ is a *number* and $|\lambda\rangle$ is a *vector*.

**Also note:** $|\lambda\rangle$ and $\lambda|\lambda\rangle$ are two *different* vectors. They point in the same direction but have different lengths.
**Example:**
- Let $O$ be a 4-dim operator with matrix representation in a particular basis given by:

$$O = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

- Then it has 4 eigenvectors given by:

$$|A\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |B\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |C\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |D\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- **Check:**

$$O|A\rangle = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 5|A\rangle$$

- **Similarly:** $O|B\rangle = 3/2|B\rangle$, $O|C\rangle = 2|C\rangle$, $O|D\rangle = -7|D\rangle$

- We say: "$|A\rangle$ is an eigenvector of $O$ with eigenvalue 5, $|B\rangle$ is another eigenvector of $O$ with eigenvalue $3/2$, etc..."