Motivating Structural Realist Interpretations of Spacetime

Jonathan Bain

Dept. of Humanities and Social Sciences
Polytechnic Institute of New York University
Brooklyn, New York

1. Realism With Respect to What?
2. Dynamical vs. Kinematical Structure
3. Is Structure Jones-Underdetermined?
4. What is Structure?
1. Realism With Respect to What?

**Scientific Realism**

Successful theories should be *interpreted literally*: we should take them at their face-value.

"Jones" Underdetermination (Jones 1991)

- Successful theories typically admit alternative mathematical formulations that disagree at the level of ontology.

- **Thus**: What should scientific realists be realists about?
1. Realism With Respect to What?

**General Relativity**

Tensor models:

\((M, g_{ab})\)

- differentiable manifold
- metric field satisfying Einstein equations

Einstein algebra (EA) models

\((\mathcal{R}^\infty, \mathcal{R}, g)\)

- commutative ring
- subring of \(\mathcal{R}^\infty\) isomorphic to \(\mathbb{R}\)
- multilinear map on space of derivations of \((\mathcal{R}^\infty, \mathcal{R})\) and its dual, satisfying Einstein equations

- **Idea:** Reconstruct \(M\) as collection of maximal ideals of commutative ring \(C^\infty(M)\) of smooth functions on \(M\).

- **Different Indivs.-based Ontologies:** points vs. ideals

- **Common Structure:** Differentiable structure
1. Realism With Respect to What?

**Claim:** Manifold points *kinematically matter*; maximal ideals do not.

**Consider:** GR with asymptotic boundary conditions.

- Asymptotically flat GR.
- GR with singularities.

**Tensor Models**

- Replace manifold $M$ with manifold with boundary $M' = M \cup \partial M$.
- $(M, g_{ab})$ is $\text{Diff}(M)$-invariant.
- $(M', g_{ab})$ is $\text{Diff}_c(M)$-invariant, but *not necessarily* $\text{Diff}(M)$-invariant.

\[ \wedge \]

$diffeomorphisms$ $on$ $M$  
$with$ $compact$ $support$  
$\approx$  
"local" $diffeomorphisms$

- No morphisms that preserve *both* $M$ and $M'$.
- $M$ and $M'$ belong to *different* categories.
1. Realism With Respect to What?

Claim: Manifold points *kinematically matter*; maximal ideals do not.

Consider: GR with asymptotic boundary conditions.

- Asymptotically flat GR.
- GR with singularities.

*Einstein Algebra (EA) Models*

1. Replace ring $\mathcal{R}^\infty \cong C^\infty(M)$ with sheaf $\mathcal{R}_{Asymp}^\infty \cong C^\infty(M')$.

2. Replace *Einstein algebra* $(\mathcal{R}^\infty, g)$ with sheaf of *Einstein algebras* $(\mathcal{R}_{Asymp}^\infty, g)$.

- $(\mathcal{R}^\infty, g)$ and $(\mathcal{R}_{Asymp}^\infty, g)$ are objects in a *single category*: the category of "structured spaces" (Heller & Sasin 1995).
- There are morphisms that preserve the structure of *both* $(\mathcal{R}^\infty, g)$ and $(\mathcal{R}_{Asymp}^\infty, g)$. 

1. Realism With Respect to What?

**Claim:** Manifold points *kinematically matter*; maximal ideals do not.

**Consider:** GR with asymptotic boundary conditions.
- Asymptotically flat GR.
- GR with singularies.

**Upshot:**
- Kinematical structure of EA models: "global" differentiable structure (morphisms preserving \((\mathcal{R}^\infty, g), (\mathcal{R}_{\text{Asymp}}^\infty, g))\).
- Kinematical structure of tensor models: "local" differentiable structure (differentiable structure at \(p\) depends on whether \(p \in M\) or \(p \in \partial M\)).
1. Realism With Respect to *What?*

**General Relativity**

**Tensor models:**

\[(M, g_{ab}^{ASD})\]

- anti-self-dual metric
- satisfying vacuum Einstein equations

**Twistor models**

\[(\mathcal{P}, \tau, \rho)\]

- "curved" twistor space
- differential forms on \(\mathcal{P}\)

- Non-linear Graviton Penrose Transformation
  (Penrose 1976)

**Idea:** Modify correspondence between Minkowski spacetime and twistor space "infinitesimally" for curved spacetimes.

**Different Indivs.-based Ontologies:** Points vs. twistors.

**Common Structure:** Conformal structure
1. Realism With Respect to What?

**General Relativity**

**Tensor models:**

\[(M, g_{ab}, (e_\mu)^a)\]

Gauge Theory of Gravity (Lansby et al 1998)

- Global tetrad field

**Geometric algebra (GA) models**

\[(\mathcal{D}, \bar{h}, \Omega)\]

- Dirac algebra
- Displacement gauge field (function) on \(\mathcal{D}\)
- Rotation gauge field (function) on \(\mathcal{D}\)

**Idea:** Impose displacement and rotation gauge invariance on a matter Lagrangian defined on \(\mathcal{D}\).

**Different Indivs.-based Ontologies:** Points vs. multivectors.

**Common Structure:** Metrical structure
2. Dynamical vs. Kinematical Structure

***Dynamically Equivalent Models of GR:***
- Tensor models *sans* b.c.'s $\cong$ EA models
- Tensor models *w/b.c.'s* $\cong$ EA models
- ASD tensor models $\cong$ Twistor models
- Tensor models *w/global tetrad fields* $\cong$ GA models

***Kinematically Distinct Models of GR:***
- Tensor models: local differentiable structure
- EA models: global differentiable structure
- Twistor models: conformal structure
- GA models: metrical structure
### 2. Dynamical vs. Kinematical Structure

<table>
<thead>
<tr>
<th>Sector</th>
<th>Models</th>
<th>Spacetime Structure</th>
<th>Dynamical Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>GR sans b.c.'s</td>
<td>tensor</td>
<td>local differentiable</td>
<td>$(M, g_{ab}) \cong (\mathcal{R}^\infty, g)$</td>
</tr>
<tr>
<td></td>
<td>EA</td>
<td>global differentiable</td>
<td></td>
</tr>
<tr>
<td>GR w/b.c.'s</td>
<td>tensor</td>
<td>local differentiable</td>
<td>$(M \cup \partial M, g_{ab}) \cong (\mathcal{R}^\infty_{Asymp}, g)$</td>
</tr>
<tr>
<td></td>
<td>EA</td>
<td>global differentiable</td>
<td></td>
</tr>
<tr>
<td>ASD-GR</td>
<td>tensor</td>
<td>local differentiable</td>
<td>$(M, g_{ab}^{ASD}) \cong (\mathcal{P}, \tau, \rho)$</td>
</tr>
<tr>
<td></td>
<td>twistor</td>
<td>conformal</td>
<td></td>
</tr>
<tr>
<td>tetrad-GR</td>
<td>tensor</td>
<td>local differentiable</td>
<td>$(M, g_{ab}, (e_\mu)^a) \cong (\mathcal{D}, \bar{h}, \Omega)$</td>
</tr>
<tr>
<td></td>
<td>GA</td>
<td>metrical</td>
<td></td>
</tr>
</tbody>
</table>
2. Dynamical vs. Kinematical Structure

*Suggests a Distinction Between:*

(A) A structural realist interpretation of a theory.  

An ontological commitment to the dynamical structure associated with the theory.

(B) A structural realist interpretation of spacetime as described by a particular formulation of a given theory.  

An interpretation of spacetime as given by the kinematical structure associated with that formulation of the theory.
3. Is Structure Jones-Underdetermined?

**Claim:** Jones Underdetermination cannot motivate structural realism.

**Why?**

Alternative formalisms disagree

(i) At the level of individuals

**AND**

(ii) At the level of structure

**THUS**

Not only are individuals-based interpretations of a single theory underdetermined; so are structural realist interpretations.
Pooley (2006, pp. 87-88):
"Consider a model of a theory of Newtonian gravitation formulated using an action-at-a-distance force and an
*empirically equivalent* model of the Newton-Cartan formulation of the theory. There is no (primitive) element of the second model which is structurally isomorphic to the flat inertial connection of the first model, and there are no (primitive) elements of the first model which are structurally isomorphic to the gravitational potential field, or the non-flat inertial structure of the second. Clearly a more sophisticated notion of structure is needed if it is to be something common to models of both formulations of the theory."
3. Is Structure Jones-Underdetermined?

But:

- Not really an example of Jones Underdetermination: Two ways of formulating the same theory in the same (tensor) formalism.

- Can a single theory admit distinct formulations in a single formalism that differ at the level of structure?
3. Is Structure Jones-Underdetermined?

I. Theories of Newtonian Gravity (NG) with a grav. potential field $\Phi$.

$$(M, h^{ab}, t_{ab}, \nabla_a, \Phi, \rho)$$

$$h^{ab} t_{ab} = 0 = \nabla_c h^{ab} = \nabla_c t_{ab} \quad \text{Orthogonality/compatibility}$$

$$h^{ab} \nabla_a \nabla_b \Phi = 4\pi G \rho \quad \text{Poisson equation}$$

$$\xi^a \nabla_a \xi^b = -h^{ab} \nabla_a \Phi \quad \text{Equation of motion}$$

---

**Ex. 1:**

Neo-Newtonian NG

$R^a_{bcd} = 0$

<table>
<thead>
<tr>
<th>Spacetime</th>
<th>Dynamical</th>
</tr>
</thead>
<tbody>
<tr>
<td>gal</td>
<td>(\overline{\text{m'a'r}})</td>
</tr>
</tbody>
</table>

**Ex. 2:**

Island Universe Neo-Newtonian NG

$R^a_{bcd} = 0, \quad \Phi \to 0 \text{ as } x^i \to \infty$

<table>
<thead>
<tr>
<th>Spacetime</th>
<th>Dynamical</th>
</tr>
</thead>
<tbody>
<tr>
<td>gal</td>
<td>gal</td>
</tr>
<tr>
<td>(\Phi \leftrightarrow \Phi + \varphi(t))</td>
<td>(\overline{\text{m'a'r}})</td>
</tr>
</tbody>
</table>

**Ex. 3:**

Maxwellian NG

$R^{ab}_{\phantom{ab}cd} = 0$

<table>
<thead>
<tr>
<th>Spacetime</th>
<th>Dynamical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\overline{\text{m'a'r}})</td>
<td>(\overline{\text{m'a'r}})</td>
</tr>
</tbody>
</table>
### Ex. 1:
Weak NCG

\[ R^{[a}_{[b} c] d] = 0 \]

<table>
<thead>
<tr>
<th>Spacetime</th>
<th>Dynamical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{\text{leib}} )</td>
<td>( \overline{\text{leib}} )</td>
</tr>
</tbody>
</table>

### Ex. 2:
Asymptotically spatially flat weak NCG

\[ R^{[a}_{[b} c] d] = 0, \quad R_{abcd} = 0 \text{ at spatial infinity} \]

<table>
<thead>
<tr>
<th>Spacetime</th>
<th>Dynamical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{gal} )</td>
<td>( \Phi \mapsto \Phi + \varphi(t) )</td>
</tr>
</tbody>
</table>

### Ex. 3:
Strong NCG

\[ R^{[a}_{[b} c] d] = 0, \quad R^{ab}_{\quad cd} = 0 \]

<table>
<thead>
<tr>
<th>Spacetime</th>
<th>Dynamical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sim \text{mar} )</td>
<td>( \sim \text{mar} )</td>
</tr>
</tbody>
</table>
3. Is Structure Jones-Underdetermined?

<table>
<thead>
<tr>
<th>Theory</th>
<th>ST symmetries</th>
<th>Dynamical symmetries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neo-Newtonian NG</td>
<td>gal</td>
<td>mar</td>
</tr>
<tr>
<td>Island Universe Neo-Newt NG</td>
<td>gal</td>
<td>gal and Φ \mapsto Φ + ϕ(t)</td>
</tr>
<tr>
<td>Maxwellian NG</td>
<td>mar</td>
<td>mar</td>
</tr>
<tr>
<td>Weak NCG</td>
<td>leib</td>
<td>leib</td>
</tr>
<tr>
<td>Asymp. spatially flat Weak NCG</td>
<td>gal</td>
<td>gal and Φ \mapsto Φ + ϕ(t)</td>
</tr>
<tr>
<td>Strong NCG</td>
<td>mar</td>
<td>mar</td>
</tr>
</tbody>
</table>

**Empirically Indistinguishable Theories**

(a) Island Universe Neo-Newt NG; Asymp. spatially flat Weak NCG  
(b) Neo-Newt NG; Max NG; Strong NCG

**Example of Underdetermination of Structure?**

Case (a)? No:

- Possess same spacetime symmetries, hence make the same ontological commitments *vis-a-vis* spacetime structure.
### 3. Is Structure Jones-Underdetermined?

<table>
<thead>
<tr>
<th>Theory</th>
<th>ST symmetries</th>
<th>Dynamical symmetries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neo-Newtonian NG</td>
<td>gal</td>
<td>ñar</td>
</tr>
<tr>
<td>Island Universe Neo-Newt NG</td>
<td>gal</td>
<td>gal and Φ ↦ Φ + ϕ((t))</td>
</tr>
<tr>
<td>Maxwellian NG</td>
<td>ñar</td>
<td>ñar</td>
</tr>
<tr>
<td>Weak NCG</td>
<td>leib</td>
<td>leib</td>
</tr>
<tr>
<td>Asymp. spatially flat Weak NCG</td>
<td>gal</td>
<td>gal and Φ ↦ Φ + ϕ((t))</td>
</tr>
<tr>
<td>Strong NCG</td>
<td>ñar</td>
<td>ñar</td>
</tr>
</tbody>
</table>

**Empirically Indistinguishable Theories**

(a) Island Universe Neo-Newt NG; Asymp. spatially flat Weak NCG
(b) Neo-Newt NG; Max NG; Strong NCG

**Example of Underdetermination of Structure?**

Case (b)?

- All *disagree* on their kinematical structure; *i.e.*, what they take to be the structure of spacetime.

- **But**: All *agree* on their dynamical structure.
3. Is Structure Jones-Underdetermined?

**Claim 1.** Structural realist interpretations of different formulations of a single theory do not suffer from underdetermination of *dynamical structure*.

**Claim 2.** Structural realist interpretations of spacetime as represented by a particular formulation of a theory are underdetermined.

**But:** Underdetermination of spacetime structure:

- Has no affect on *current* empirical adequacy of the theory.
- Is susceptible to *future* empirical tests:

  **Extensions of GR to Quantum Gravity:**
  Twistors $\Rightarrow$ Twistor approach to QG.
  Einstein algebras $\Rightarrow$ Heller & Sasin (1999) QG.
  Geometric algebra $\Rightarrow$ Background dependent QG.
4. What is Structure?

*Radic al Ontic Structural Realism* (French & Ladyman 2003)

Structure consists of relations devoid of *relata.*

**Untenable?**

*Set-theoretically,* perhaps so.

- Suppose *structure = isomorphism class of structured sets = [{X, R_i}].*

- A (binary) *relation R on X* is a subset of $X \times X,$ the set of all ordered pairs $(x_1, x_2),$ $x_1, x_2 \in X.$

- An *ordered pair* $(x_1, x_2)$ is the set $\{x_1, \{x_1, x_2\}\}.$

- Ineliminable reference to elements ("*relata*") of a set.
4. What is Structure?

**Radical Ontic Structural Realism** (French & Ladyman 2003)
Structure consists of relations devoid of *relata*.

**Untenable?**

*Category-theoretically*, perhaps not.

- Suppose *structure = object in a category*.
- "Internal" constituents of an object ("elements") referred to purely in terms of "external" objects and morphisms.
4. What is Structure?

- An object 1 of a category $\mathcal{C}$ is a *terminal object* of $\mathcal{C}$ if for each object $X$ of $\mathcal{C}$, there is exactly one $\mathcal{C}$-morphism $X \to 1$.
- An *element* of an object $A$ in a category $\mathcal{C}$ is a morphism $1 \to A$, where $1$ is the terminal object in $\mathcal{C}$.

**Set Theory**

*Primitives:* sets, $\in$

\[
\begin{align*}
A & \quad \bullet \quad x_1 \\
x_1 \in A
\end{align*}
\]

**Category Theory**

*Primitives:* objects, morphisms

\[
\begin{align*}
1 & \quad \xrightarrow{x_1} \quad A
\end{align*}
\]
4. What is Structure?

- The *Cartesian product* of an object $X$ with itself is an object $P$, together with a pair of morphisms $p_1 : P \to X$, $p_2 : P \to X$ such that, for any arbitrary object $T$ with morphisms $f_1 : T \to X$, $f_2 : T \to X$, there is exactly one morphism $f : T \to P$ for which $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.

- External probe $T$, $f_1$, $f_2$, $f$ encodes internal pair structure of $P$. 
4. What is Structure?

**Objection:** Elimination of relata in name only.

- Where set theory sees "elements", category theory sees "morphisms from the terminal object".
- "No relations without relata" becomes "No objects without morphisms".

**Response**

- Manifold points have correlates in EA, but ultimately these correlates are surplus in EA models of GR.
- Similarly, set-theoretic relata have correlates in category theory, but ultimately these correlates are surplus.
  - Category-theoretic objects need not be structured sets.
  - Such objects have roles to play in articulating relevant notions of structure in physics. Baez (2006)
4. What is Structure?

What the Category-theoretic Radical Ontic Structural Realist must do:

• Provide rationale for fundamentality of category theory over set theory. (Pedroso 2008)

• Provide category-theoretic formulations of scientific theories that do not presuppose Set. (Döring & Isham 2008; Isham and Butterfield 2000; Baez 2006)

• Identify the relevant notion of structure in category-theoretic terms.
  ○ Distinguish between kinematical structure and dynamical structure in category-theoretic terms.
4. What is Structure?

**How to do physics in category theory:** (Baez 2006)

Given a theory $T$,

- "Kinematics" of $T = \text{objects } A, B, \ldots$ in category $\mathcal{C}$.
- Dynamics of $T = \text{morphisms } f: A \rightarrow B, g: C \rightarrow D, \ldots$ in $\mathcal{C}$.

**Ex1: Classical physics**

$\mathcal{C} = \text{Symp}$

objects = symplectic manifolds (classical phase spaces)
morphisms = symplectic transformations

**Ex2: Quantum physics**

$\mathcal{C} = \text{Hilb}$

objects = Hilbert spaces (quantum phase spaces)
morphisms = bounded linear operators
4. What is Structure?

How to do physics in category theory: (Baez 2006)

Given a theory $T$,

- "Kinematics" of $T = \text{objects } A, B, ... \text{ in category } \mathcal{C}$.  
- Dynamics of $T = \text{morphisms } f: A \to B, g: C \to D, ... \text{ in } \mathcal{C}$.  

However:

- The "kinematics" here describes space of \textit{dynamically} possible states.
- Distinction between \textit{kinematically} possible states and \textit{dynamically} possible states.
4. What is Structure?

*How to do field-theoretic physics:*  (Belot 2007)

A field theory consists of \((\mathcal{K}, \Delta)\), where

(i) \(\mathcal{K}\) is the space of *kinematically possible* fields \(\phi : M \to W\), where \(M\) is a differentiable manifold (viz., spacetime) and \(W\) is an appropriate space in which the fields take values.

(ii) \(\Delta\) is a set of differential equations consisting of *independent* variables (parametrizing \(M\)) and *dependent* variables (parametrizing \(W\)).

- Define space of *dynamically possible* fields \(S = \{\phi_0 \in \mathcal{K} : \phi_0\) is a solution of \(\Delta\}\).
- *Dynamical structure* = Structure of \(S\).
- *Kinematical structure* = Structure of independent variables in \(\Delta\).
4. What is Structure?

**Kinematically Distinct Models of GR:**

(a) Tensor models: local differentiable structure
(b) EA models: global differentiable structure
(c) Twistor models: conformal structure
(d) GA models: metrical structure

**Category-theoretic translations:**

(a) (i) $\text{Man} = \text{category of smooth manifolds}$
    (ii) $\text{Manb} = \text{category of smooth manifolds with boundary}$
(b) $\text{Struc} = \text{category of structured spaces}$ (Heller and Sasin 1995)
(c) $\text{Twist} = \text{category of (curved) twistor spaces}$
(d) $\text{Cliff}_{(1,3)} = \text{category of Dirac algebras}$
4. What is Structure?

<table>
<thead>
<tr>
<th>Sector</th>
<th>Models</th>
<th>Spacetime Structure</th>
<th>Dynamical Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>GR <em>sans</em> b.c.'s</td>
<td>tensor</td>
<td>local differentiable</td>
<td>Man $(M, g_{ab}) \cong (M, g_{ab})$</td>
</tr>
<tr>
<td></td>
<td>EA</td>
<td>global differentiable</td>
<td>Struc $(R^\infty, g)$</td>
</tr>
<tr>
<td>GR <em>w/b.c.'s</em></td>
<td>tensor</td>
<td>local differentiable</td>
<td>Manb $(M \cup \partial M, g_{ab}) \cong (R^\infty_{Asymp}, g)$</td>
</tr>
<tr>
<td></td>
<td>EA</td>
<td>global differentiable</td>
<td>Struc $\cong (R^\infty_{Asymp}, g)$</td>
</tr>
<tr>
<td>ASD-GR</td>
<td>tensor</td>
<td>local differentiable</td>
<td>Man $(M, g^\text{ASD}<em>{ab}) \cong (M, g^\text{ASD}</em>{ab})$</td>
</tr>
<tr>
<td></td>
<td>twistor</td>
<td>conformal</td>
<td>Twist $(\mathcal{P}, \tau, \rho)$</td>
</tr>
<tr>
<td>tetrad-GR</td>
<td>tensor</td>
<td>local differentiable</td>
<td>Man $(M, g_{ab}, (e_\mu^a)^a) \cong (D, \bar{h}, \Omega)$</td>
</tr>
<tr>
<td></td>
<td>GA</td>
<td>metrical</td>
<td>Clif$_{(1,3)}$ $(D, \bar{h}, \Omega)$</td>
</tr>
</tbody>
</table>

- Symp $\supset$ Symp$_i \cong S$ for given $(\mathcal{K}, \Delta)$. 


5. Conclusion

- Dynamical vs. kinematical structure.
- Motivates distinction between structural realist interpretations of a theory vs. structural realist interpretations of spacetime as described by a theory.
- Blunts Jones Underdetermination arguments against structural realism.
- Can be articulated in category-theoretic terms.