What Explains the Spin-Statistics Connection?

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2. The Spin-Statistics Theorem Does Not Explain SSC.
3. What Would Explain SSC?
4. Conclusion.

**Spin-statistics connection (SSC):**

i. States that obey B-E statistics possess integer spin.
ii. States that obey F-D statistics possess half-integer spin.

**Statistics in terms of particle states:**

- **"Bosonic" multiparticle state**
  - State invariant under particle permutations.
  - Require: \([a(p), a^\dagger(p')]_− = \delta(p - p')\)

- **"Fermionic" multiparticle state**
  - State invariant under particle permutations, and such that no two particles have exactly the same properties.
  - Require: \([a(p), a^\dagger(p')]_+ = \delta(p - p')\)

**Spin-statistics connection (SSC):**

i. States that obey B-E statistics possess integer spin.

ii. States that obey F-D statistics possess half-integer spin.

**Statistics in terms of field states:**

- \([\phi^\dagger(x), \phi(x')]_\pm = 0\), for spacelike \((x - x')\).
  
  \textit{(Relativistic Local Commutativity)}.

- \([\phi^\dagger(x, t), \phi(x', t)]_\pm = 0\), for \((x - x') \neq 0\)

  \textit{(Non-relativistic Local Commutativity)}.

**Spin-statistics connection (SSC):**

i. States that obey B-E statistics possess integer spin.

ii. States that obey F-D statistics possess half-integer spin.

**Spin in terms of group representations:**

- Relativistic integer (half-integer) spin state = true (double-valued) representation of Poincaré group $\mathcal{P}$.
- Non-relativistic integer (half-integer) spin state = true (double-valued) representation of Galilei group $\mathcal{G}$.

- $\mathcal{P} = SO(1,3) \rtimes \mathbb{R}^{1,3}$.
- $\mathcal{G} = (SO(3) \rtimes \mathbb{R}^3) \rtimes (\mathbb{R}^1 \times \mathbb{R}^3)$. 

- A "profound impact" in non-relativistic quantum mechanics (NQM) and non-relativistic quantum field theory (NQFT)...

"From the microscopic structure of atoms to the macroscopic structure of neutron stars, a dazzling wealth of physical phenomena would be incomprehensible without this spin-statistics rule. Many elements of condensed matter physics, for instance, band structure, Fermi liquid theory, superfluidity, superconductivity, quantum Hall effect, and so on and so forth, are consequences of this rule." (Zee 2010.)

...whose explanation had to wait until relativistic quantum field theory (RQFT)...

"...the explanation of the spin-statistics connection by Fierz and by Pauli in the late 1930s, and by Luders and Zumino and by Burgoyne in the late 1950s, ranks as one of the great triumphs of relativistic quantum field theory." (Zee 2010.)

"[The Spin-Statistics theorem]... clarifies one of the great mysteries of non-relativistic quantum theory: the contrasting symmetry properties of the wavefunctions of particles of integer (boson) versus half-integer (fermionic) spin." (Duncan 2012.)

- ... or did it?

"The spin-statistics connection seems crucial to understanding the behavior of several physical systems for which relativistic considerations seem quite insignificant... Non-relativistic theories seem to adequately describe most of these systems and the spin-statistics connection has to be inserted 'by hand' when formulating these theories." (Shaji 2009.)

"An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity... we have not been able to find a way of reproducing his arguments on an elementary level. This probably means we do not have a complete understanding of the fundamental principle involved." (Feynman 1965.)

*Spin-Statistics Theorem:* Any state of a physical system described by an RQFT must possess SSC.

**Question #1:**
Does the Spin-Statistics Theorem explain SSC in RQFTs?

**Question #2:**
What explains SSC in NQM and NQFTs?
2. The Spin-Statistics Theorem Does Not Explain SSC.

A D-N Explanation of SSC?

- D-N explains by virtue of a derivation from a set of first principles or covering laws.

- **Problem:** There is no unique set of first principles from which SSC can be derived in RQFTs.
## 2. The Spin-Statistics Theorem Does Not Explain SSC.

No unique set of first principles from which SSC can be derived in RQFTs.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Principles</th>
<th>Derived Property</th>
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| Wightman (Luders & Zumino 1958; Burgoyne 1958) | (a) RLI for fields.  
(b) SC for fields.  
(c) LC for fields. | SSC for field states. |
| Weinberg (Weinberg 1964) | (a) RLI for $S$-matrix.  
(b) CD for $S$-matrix. | SSC for particle states. |
| Lagrangian (Fierz 1939; Pauli 1940) | (a) RLI for fields.  
(b) SC for fields.  
(c) Causality.  
|                        | (a) RLI for fields.  
(b) Causality. | SSC for fermionic field states.  
|                        | | SSC for bosonic field states. |
| Algebraic (Guido & Longo 1995) | (a) Modular covariance.  
(b) Additivity.  
(c) "Algebraic causality". | SSC for DHR "particle" representations. |
2. The Spin-Statistics Theorem Does Not Explain SSC.

A Unifying Explanation of SSC?

- A unifying explanation explains by virtue of being an instantiation of a general, stringent, and unified argument pattern endorsed by a consensus of a particular community.

- **Problem**: There is no consensus on which approach to RQFTs should be adopted.
Which Approach?

(I) Axiomatic/algebraic ("purist") approaches?

Problem of Empirical Import:
No realistic interacting models of the relevant axioms.

Qualifications

- Realistic interacting model = model for a 4-dim RQFT (like QED and QCD) from which predictions have been derived and confirmed.
- Non-interacting models exist.
- Unrealistic interacting models exist: (Rivasseau 2003)
  - $P(\phi)_2$, $\phi^4_3$, Yukawa model (2-dim, 3-dim), Gross-Neveu model (2-dim).
2. The Spin-Statistics Theorem Does Not Explain SSC.

Which Approach?

(II) Weinberg/Lagrangian ("pragmatist") approaches?

**Renormalization Problem**: $S$-matrix assumes non-interacting multi-particle states at asymptotic times are related to interacting multi-particle states at finite times; and this requires introduction of infinitely renormalized parameters.

**$\text{UV Problem}$**: For many realistic interacting QFTs, the power series expansion of the $S$-matrix contains divergent terms at high energies.

**Convergence Problem**: For many realistic interacting QFTs, the power series expansion of the $S$-matrix may not converge.
2. The Spin-Statistics Theorem Does Not Explain SSC.

Qualifications

- Renormalization, UV, Convergence Problems are common to any approach that employs renormalized perturbation theory to derive predictions from most realistic interacting RQFTs.

- Some realistic interacting RQFTs (e.g., QCD) do not suffer the UV Problem.

- Renormalization Group techniques address Renormalization and UV Problems.
2. The Spin-Statistics Theorem Does Not Explain SSC.

A Causal Explanation of SSC?

- A causal explanation (of a "general" type of event/fact) explains by virtue of specifying possible causal histories.

"A general causal explanation says what the causal histories of instances of the event-type being explained have in common, or says something about what it would have taken for a given alternative type of event to have occurred instead, which applies to many or most of the instances of the event-type being explained." (Skow 2013.)

- **Gloss:** A general causal explanation explains by virtue of placing constraints on *dynamically* possible states.
2. The Spin-Statistics Theorem Does Not Explain SSC.

A Causal Explanation of SSC?

- **Problem:** The Spin-Statistics theorem explains (to the extent that it does explain) by virtue of placing a constraint on \textit{kinematically} possible states.
  
  - Any state of a physical system described by an RQFT must possess SSC, \textit{regardless} of what dynamics it satisfies.
2. The Spin-Statistics Theorem Does Not Explain SSC.

A Structural Explanation of SSC?

- The Spin-Statistics theorem demonstrates how a set of principles limits the kinematically possible states of physical systems to those that possess SSC.

"...a structural explanation can be understood as one in which the *explanandum* is explained by showing how the (typically mathematical) structure of the theory itself limits what sort of objects, properties, states, or behaviors are admissible within the frame-work of that theory, and then showing that the *explanandum* is in fact a consequence of that structure." (Bokulich 2011.)
2. The Spin-Statistics Theorem Does Not Explain SSC.

A Structural Explanation of SSC?

Problems:

- Should the set of principles be taken to represent real physical structures? (Bueno & French 2012.)
  - *But:* Which structures? No unique set of principles.

- SSC fundamental to explanations in NQM and NQFTs:
  - *Electronic structure of solids (NQM).*
  - *Formation of white dwarves (NQM).*
  - *Formation of Cooper pairs in superconductors and Bose-Einstein condensates (NQFTs).*

- How does a structural explanation of SSC in RQFTs explain SSC in NQM and NQFTs?
2. The Spin-Statistics Theorem Does Not Explain SSC.

**General Concern:**

- Spin-Statistics theorem (*both* purist & pragmatist versions) demonstrates that SSC is an essential property of *non-interacting*, and at most *unrealistic interacting* RQFTs.

- In NQM and NQFTs, SSC appears as a (brute fact) property of realistic interacting theories:
  - *Electronic structure of solids* (*interacting NQM*).
  - *Formation of white dwarves* (*interacting NQM*).
  - *Formation of Cooper pairs in superconductors and Bose-Einstein condensates* (*interacting NQFTs*).

- If the explanandum is *SSC in realistic interacting theories*, then the Spin-Statistics theorem by itself does not provide an explanation.
3. What Would Explain SSC?

Weatherall's (2011) example.

- **General fact**: \(m_i = m_g\).
- **Relevant theories**: Newtonian gravitation and GR.

"The explanatory demand is to show how, given some superseding theory, a general fact as expressed within one theory is really necessary or to be expected within the regime in which the old theory is successful... The explanatory work, then, is done by presenting the details of the relationship between the two theories." (Weatherall 2011.)

Analogously...

- **General fact**: SSC in realistic interacting theories.
- **Relevant theories**: NQM/NQFTs and RQFTs.
3. What Would Explain SSC?

Continuing the Analogy:

• \( m_i = m_g \) cannot be expressed in GR, but can be shown to arise in the limit as one goes from GR to Newton-Cartan theory (via lightcone flattening), and then is carried over to Newtonian gravitation (via Trautman's theorem).

• **Analogously:** SSC cannot be expressed in realistic interacting RQFTs, but can be shown to arise in the limit as one goes from realistic interacting RQFTs to non-interacting RQFTs, and then is carried over to realistic interacting NQFTs/NQM.

• **Example:** Consider how carry-over from non-interacting RQFTs to realistic interacting NQFTs can be represented...
3. What Would Explain SSC?

\[
\begin{align*}
\text{(relativistic integer/} & \quad \text{SSC} & \quad \text{(relativistic bosonic/} \\
\text{half-integer spin state)} & \quad \text{essential} & \quad \text{fermionic state)} \\
\downarrow \text{non-relativistic limit} & & \downarrow \text{non-relativistic limit} \\
\text{(non-relativistic integer/} & \quad \text{SSC} & \quad \text{(non-relativistic bosonic/} \\
\text{half-integer spin state)} & \quad \text{brute fact} & \quad \text{fermionic state)}
\end{align*}
\]
3. What Would Explain SSC?

\[ \begin{align*}
\text{true/double-valued rep} & \quad \text{of Poincaré group } \mathcal{P} \\
& \quad \text{non-relativistic limit}
\end{align*} \quad \text{SSC} \quad \begin{align*}
\text{relativistic bosonic/fermionic state} \\
& \quad \text{non-relativistic limit}
\end{align*} \]

\[ \begin{align*}
\text{true/double-valued rep} & \quad \text{of Galilei group } \mathcal{G} \\
& \quad \text{brute fact}
\end{align*} \quad \text{SSC} \quad \begin{align*}
\text{non-relativistic bosonic/fermionic state}
\end{align*} \]
3. What Would Explain SSC?

"Speed-space" contraction of \( \mathcal{P} \) yields \( \mathcal{G} \): (Bacry & Lévy-Leblond 1969)

Hold fixed \( J, H \), replace \( K, P \) with \( \varepsilon K, \varepsilon P \), take limit \( \varepsilon \to 0 \) of Lie brackets.

- Small velocities (\( K \)) \( (v/c \to 0) \).
- Small spatial translations (\( P \))
  - spacelike intervals \( \ll \) timelike intervals
  - spacelike intervals \( \to \) equal time spatial intervals
3. What Would Explain SSC?

\[
\begin{align*}
\text{true/double-valued rep} & \quad \text{SSC} & \quad \text{relativistic local commutativity} \\
of \text{Poincaré group } \mathcal{P} & & \\
\downarrow \text{speed-space contraction} & & \\
\text{true/double-valued rep} & \quad \text{SSC} & \quad \text{non-relativistic local commutativity} \\
of \text{Galilei group } \mathcal{G} & & \\
\downarrow \text{speed-space contraction} & & \\
\end{align*}
\]

"Speed-space" contraction of \( \mathcal{P} \) yields \( \mathcal{G} \): (Bacry & Lévy-Leblond 1968)

Hold fixed \( \mathbf{J}, \mathbf{H} \), replace \( \mathbf{K}, \mathbf{P} \) with \( \varepsilon \mathbf{K}, \varepsilon \mathbf{P} \), take limit \( \varepsilon \to 0 \) of Lie brackets.

- Small velocities (\( \mathbf{K} \)) (\( v/c \to 0 \)).
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  - spacelike intervals \( \to \) equal time spatial intervals
3. What Would Explain SSC?

- A *kinematic* result: More than one non-relativistic limit may be available, depending on form of dynamics. (Brown & Holland 2003.)
- There are true/double-valued reps of $G$ that describe realistic interacting NQFTs.
4. Conclusion.

- SSC = a general observational feature of the world.
- Spin-Statistics theorem does not explain it.
- Spin-Statistics theorem coupled with the appropriate intertheoretic relations among RQFTs, NQFTs, and NQM would.
References.


